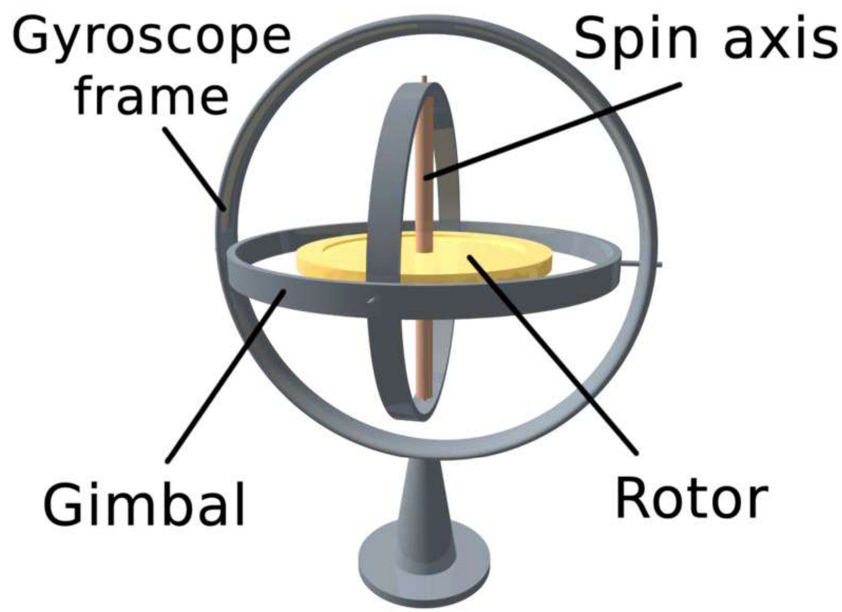
Gyroscope

- A gyroscope is a device for measuring or maintaining orientation, based on the principles of angular momentum.
- Mechanically, a gyroscope is a spinning wheel or disk in which the axle is free to assume any orientation. Although this orientation does not remain fixed, it changes in response to an external torque much less and in a different direction than it would without the large angular momentum associated with the disk's high rate of spin and moment of inertia.

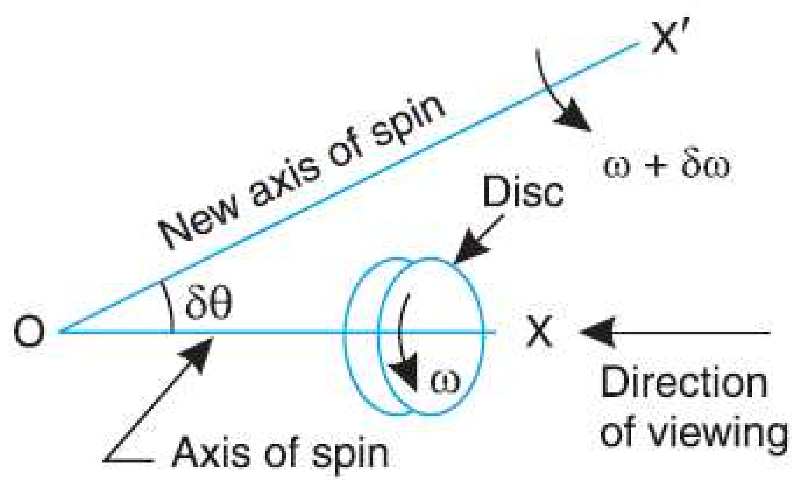
Since external torque is minimized by mounting the device in gimbals, its orientation remains nearly fixed, regardless of any motion of the platform on which it is mounted.



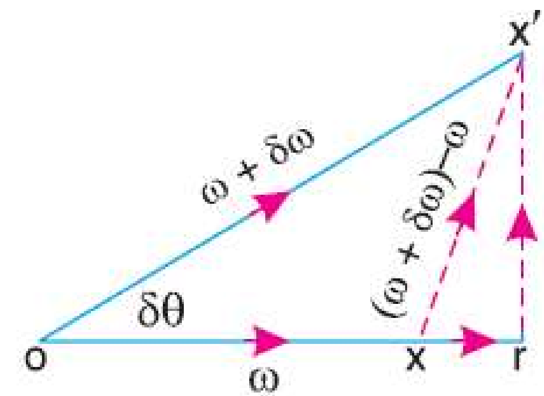
Precessional Angular Motion

(Vectorial representation of angular motion)

- We know that the angular acceleration is the rate of change of angular velocity with respect to time.
- It is a vector quantity and may be represented by drawing a vector diagram with the help of right hand screw rule.



(a)



(b)

- Consider a disc, as shown in Fig. (*a), revolving or spinning about the axis OX (known as* axis of spin) in anticlockwise when seen from the front, with an angular velocity in a plane at right angles to the paper.

- After a short interval of time *t, let the disc be spinning about the new axis of spin OX' (at an* angle $\delta\theta$) with an angular velocity (ω +$\delta\omega$).
- Using the right hand screw rule, initial angular velocity of the disc ω is represented by vector *ox; and the final angular velocity of the disc (ω +$\delta\omega$) is* represented by vector *ox' as shown in Fig. (b).*
- *The vector xx' represents the change of angular* velocity in time δt *i.e. the angular acceleration of the disc. This may be resolved into two components,* one parallel to *ox and the other perpendicular to ox.*

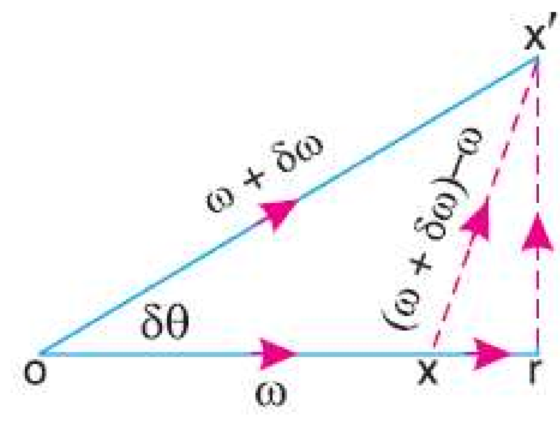
Component of angular acceleration in the direction of ox,

$$\alpha_t = \frac{xr}{\delta t} = \frac{or - ox}{\delta t} = \frac{ox' \cos \delta\theta - ox}{\delta t}$$

$$= \frac{(\omega + \delta\omega) \cos \delta\theta - \omega}{\delta t} = \frac{\omega \cos \delta\theta + \delta\omega \cos \delta\theta - \omega}{\delta t}$$

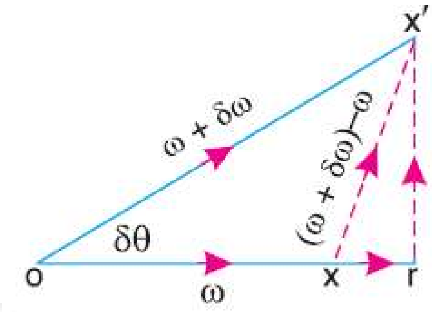
Since $\delta\theta$ is very small, therefore substituting $\cos \delta\theta = 1$, we have

$$\alpha_t = \frac{\omega + \delta\omega - \omega}{\delta t} = \frac{\delta\omega}{\delta t}$$



(b)

In the limit, when $\delta t \to 0$,



$$\alpha_t = \underset{\delta t \to 0}{\mathrm{Lt}} \left(\frac{\delta \omega}{\delta t} \right) = \frac{d\omega}{dt}$$

Component of angular acceleration in the direction perpendicular to ox,

$$\alpha_c = \frac{rx'}{\delta t} = \frac{ox' \sin \delta\theta}{\delta t} = \frac{(\omega + \delta\omega)\sin\delta\theta}{\delta t} = \frac{\omega \sin\delta\theta + \delta\omega . \sin\delta\theta}{\delta t}$$

Since $\delta\theta$ in very small, therefore substituting $\sin\delta\theta = \delta\theta$, we have

$$\alpha_c = \frac{\omega . \delta\theta + \delta\omega . \delta\theta}{\delta t} = \frac{\omega . \delta\theta}{\delta t}$$

...(Neglecting $\delta\omega.\delta\theta$, being very small)

In the limit when $\delta t \to 0$,

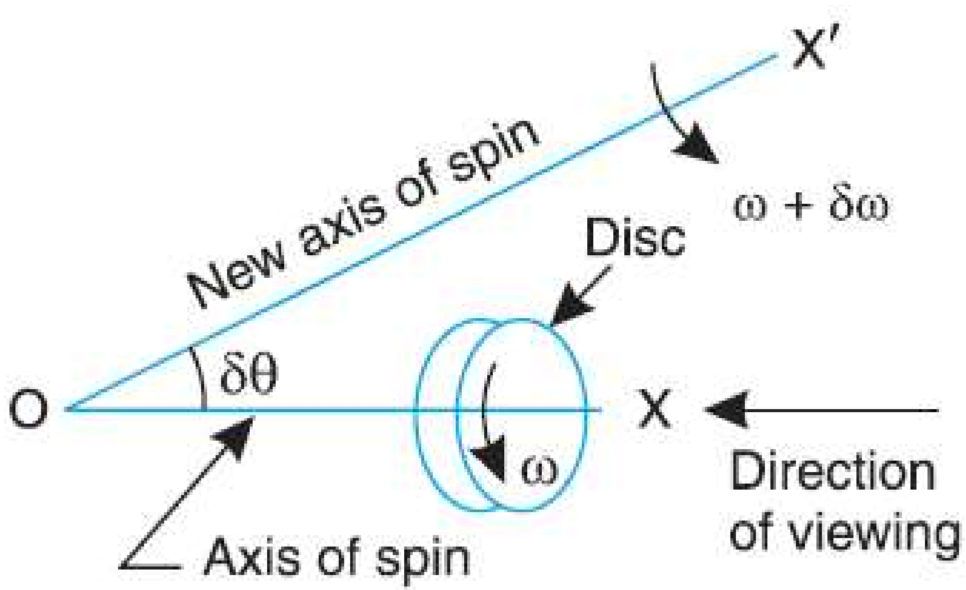
$$\alpha_c = \underset{\delta t \to 0}{\mathrm{Lt}} \frac{\omega . \delta\theta}{\delta t} = \omega \times \frac{d\theta}{dt} = \omega . \omega_P \qquad \ldots \left(\text{Substituting } \frac{d\theta}{dt} = \omega_P \right)$$

$\therefore$ Total angular acceleration of the disc

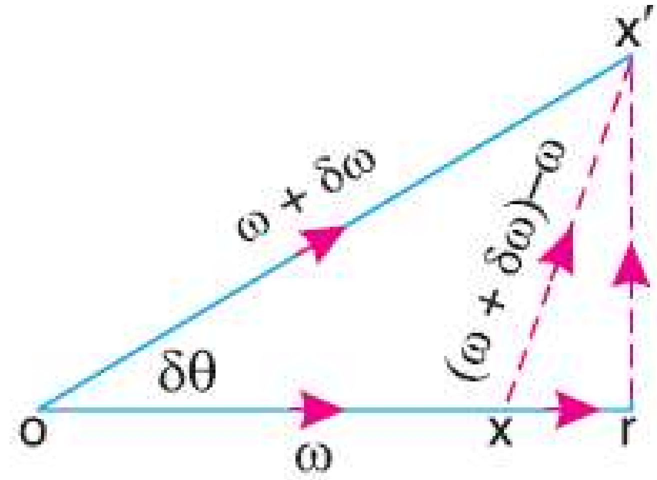
$$= \text{vector } xx' = \text{vector sum of } \alpha_t \text{ and } \alpha_c$$

$$= \frac{d\omega}{dt} + \omega \times \frac{d\theta}{dt} = \frac{d\omega}{dt} + \omega . \omega_P$$

- where *$d\theta/dt$ is the angular velocity of the axis of spin about a certain axis, which is perpendicular to* the plane in which the axis of spin is going to rotate.



(a)



(b)

where $d\theta/dt$ is the angular velocity of the axis of spin about a certain axis, which is perpendicular to the plane in which the axis of spin is going to rotate. This angular velocity of the axis of spin (*i.e.* $d\theta/dt$) is known as ***angular velocity of precession*** and is denoted by ω_P. The axis, about which the axis of spin is to turn, is known as ***axis of precession.*** The angular motion of the axis of spin about the axis of precession is known as ***precessional angular motion.***

Notes: 1. The axis of precession is perpendicular to the plane in which the axis of spin is going to rotate.

2. If the angular velocity of the disc remains constant at all positions of the axis of spin, then $d\theta/dt$ is zero; and thus α_c is zero.

3. If the angular velocity of the disc changes the direction, but remains constant in magnitude, then angular acceleration of the disc is given by

$$\alpha_c = \omega . d\theta/dt = \omega . \omega_P$$

The angular acceleration α_c is known as ***gyroscopic acceleration.***

Gyroscopic Couple

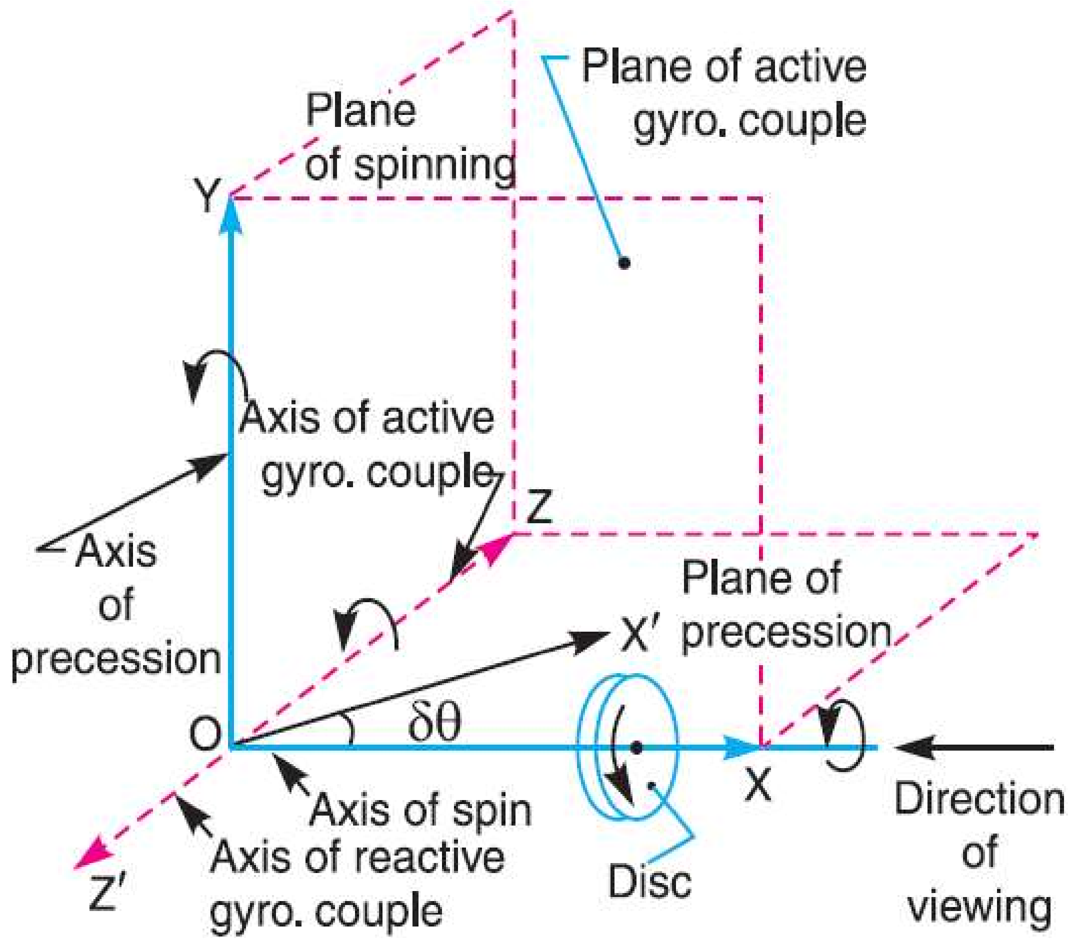
- Consider a disc spinning with an angular velocity ω rad/s about the axis of spin *OX, in anticlockwise direction when seen from the front, as shown in Fig.*
- *Since the plane in which the disc is rotating is parallel to the plane YOZ therefore it is called plane of spinning.*

The plane XOZ is a horizontal plane and the axis of spin rotates in a plane parallel to the horizontal plane about an axis OY.

In other words, the axis of spin is said to be rotating or processing about an axis OY.

In other words, the axis of spin is said to be rotating about an axis OY (which is perpendicular to both the axes OX and OZ) at an angular velocity ω_p rad/s.

This horizontal plane XOZ is called plane of precession and OY is the axis of precession.



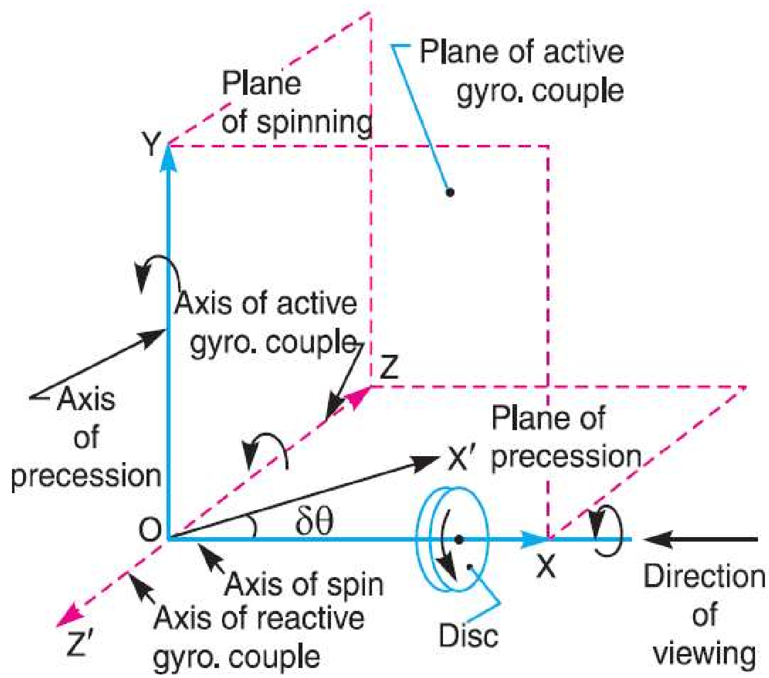
Let I = Mass moment of inertia of the disc about OX, and

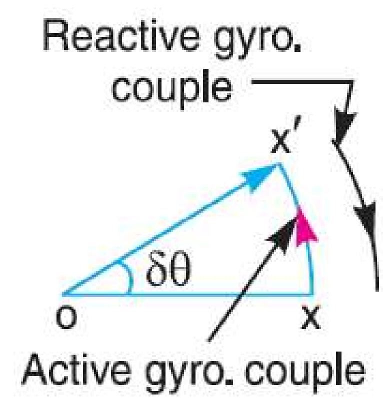
ω = Angular velocity of the disc.

$\therefore$ Angular momentum of the disc

$= I.\omega$

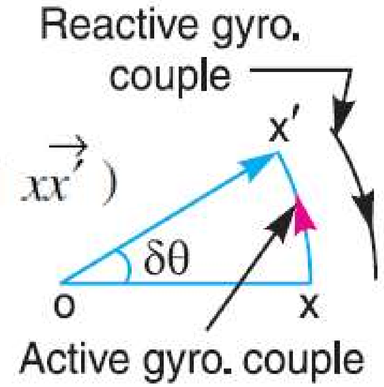
- Since the angular momentum is a vector quantity, therefore it may be represented by the vector OX , *as shown in Fig.*
- *The axis of spin OX is also rotating anticlockwise when seen from the top about the axis OY.*
- *Let the axis OX is turned in the plane XOZ through a small angle $\delta\theta$ radians to the position OX', in time δt seconds. Assuming the angular velocity ω to be constant, the angular momentum will now be represented by vector OX'.*





∴ Change in angular momentum

$$= \vec{ox'} - \vec{ox} = \vec{xx'} = \vec{ox}.\delta\theta$$
$$= I.\ \omega.\delta\theta \qquad \text{...(in the direction of } \vec{xx'}\text{)}$$



and rate of change of angular momentum

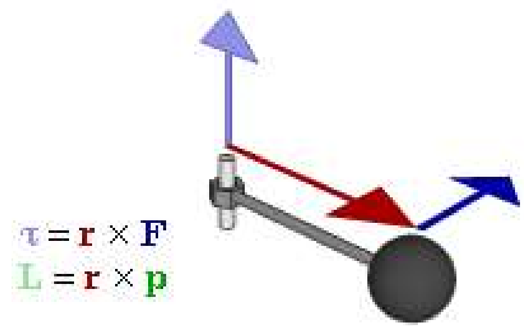
$$= I.\omega \times \frac{\delta\theta}{dt}$$

Since the rate of change of angular momentum will result by the application of a couple to the disc, therefore the couple applied to the disc causing precession,

$$C = \underset{\delta t \to 0}{\mathrm{Lt}}\ I.\omega \times \frac{\delta\theta}{\delta t} = I.\omega \times \frac{d\theta}{dt} = I.\omega.\omega_P \qquad \ldots\left(\because \frac{d\theta}{dt} = \omega_P\right)$$

where ω_p = Angular velocity of precession of the axis of spin or the speed of rotation of the axis of spin about the axis of precession *OY.*

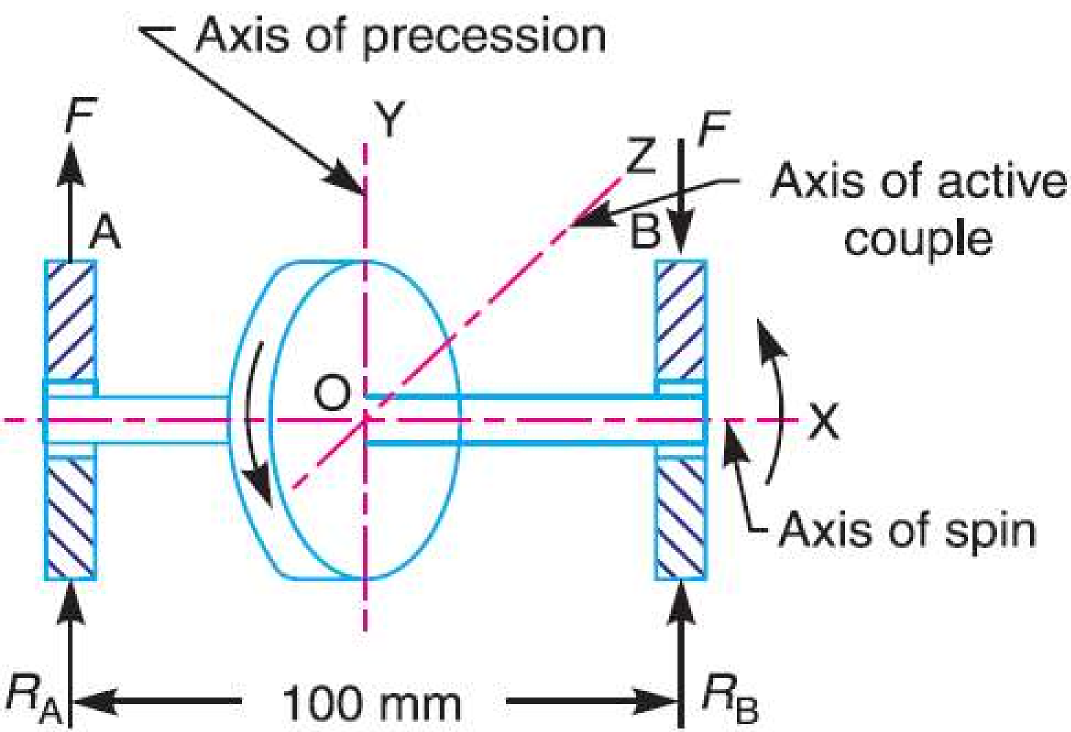
In S.I. units, the units of *C is N-m when I is* in kg-m^2.

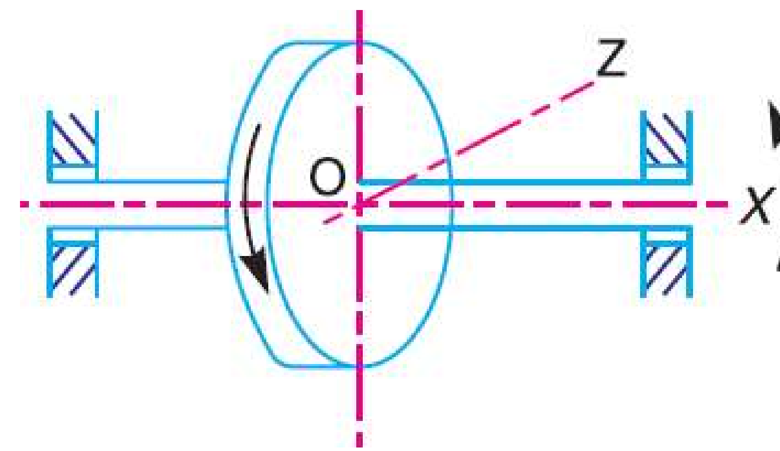


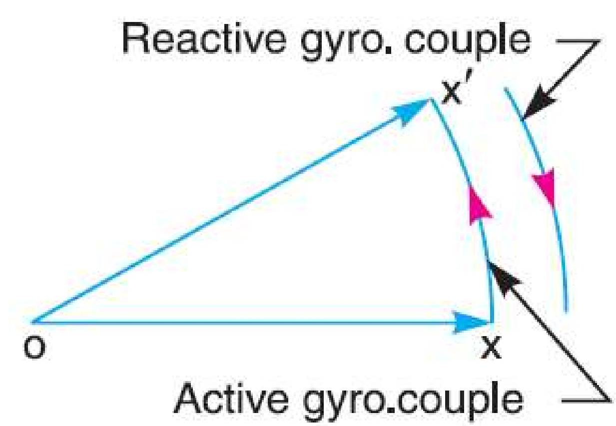
Relationship between force (F), torque (τ), momentum (p), and angular momentum (L) vectors in a rotating system, r= Position vector

- *A uniform disc of 150 mm diameter has a mass of 5 kg. It is mounted centrally in bearings which maintain its axle in a horizontal plane. The disc spins about it axle with a constant speed of 1000 r.p.m. while the axle precesses uniformly about the vertical at 60 r.p.m. The directions of rotation are as shown in Fig. 14.3. If the distance between the bearings is 100 mm, find the resultant reaction at each bearing due to the mass and gyroscopic effects.*

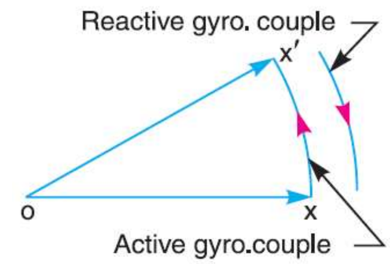
Solution. Given: d = 150 mm or r = 75 mm = 0.075 m ; m = 5 kg ; N = 1000 r.p.m. or $\omega = 2\pi \times 1000/60 = 104.7$ rad/s (anticlockwise); N_P = 60 r.p.m. or $\omega_P = 2\pi \times 60/60 = 6.284$ rad/s (anticlockwise); x = 100 mm = 0.1 m







We know that mass moment of inertia of the disc, about an axis through its centre of gravity and perpendicular to the plane of disc,



$$I = m.r^2/2 = 5\,(0.075)^2/2 = 0.014 \text{ kg m}^2$$

$\therefore$ Gyroscopic couple acting on the disc,

$$C = I.\,\omega.\,\omega_P = 0.014 \times 104.7 \times 6.284 = 9.2 \text{ N-m}$$

The direction of the reactive gyroscopic couple is shown in Fig.14.4 (*b*). Let F be the force at each bearing due to the gyroscopic couple.

$$\therefore \quad F = C/x = 9.2/0.1 = 92 \text{ N}$$

The force F will act in opposite directions at the bearings as shown in Fig. 14.4 (*a*). Now let R_A and R_B be the reaction at the bearing A and B respectively due to the weight of the disc. Since the disc is mounted centrally in bearings, therefore,

$$R_A = R_B = 5/2 = 2.5 \text{ kg} = 2.5 \times 9.81 = 24.5 \text{ N}$$

Resultant reaction at each bearing

Let R_{A1} and R_{B1} = Resultant reaction at the bearings A and B respectively.

Since the reactive gyroscopic couple acts in clockwise direction when seen from the front, therefore its effect is to increase the reaction on the left hand side bearing (*i.e.* A) and to decrease the reaction on the right hand side bearing (*i.e.* B).

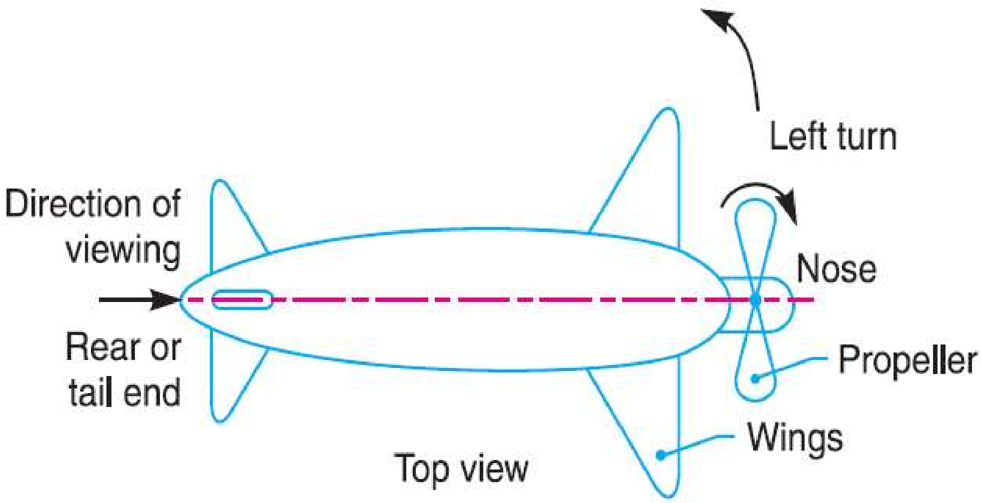
$$\therefore \quad R_{A1} = F + R_A = 92 + 24.5 = 116.5 \text{ N (upwards)} \quad \textbf{Ans.}$$

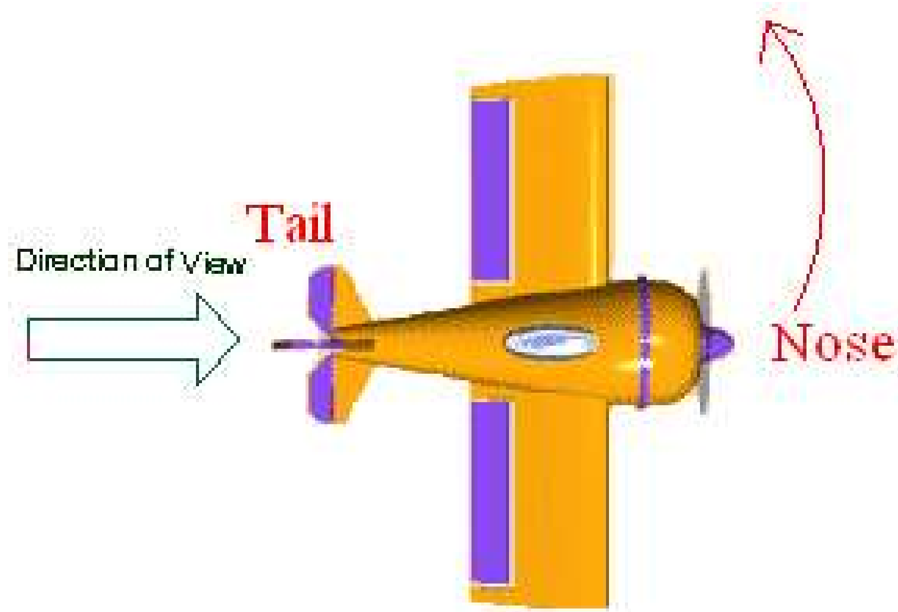
and

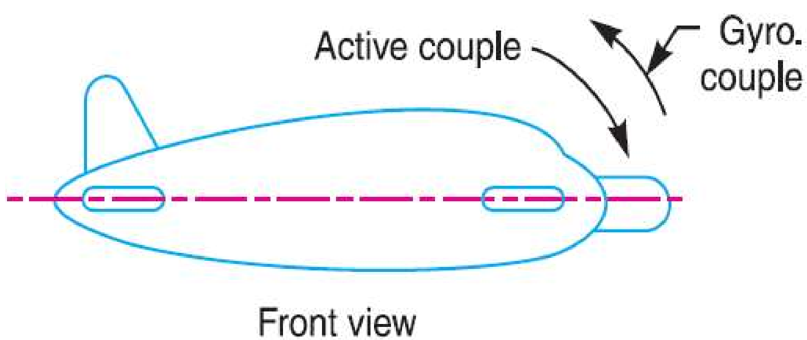
$$R_{B1} = F - R_B = 92 - 24.5 = 67.5 \text{ N (downwards)} \quad \textbf{Ans.}$$

Effect of the Gyroscopic Couple on an Aero-plane

- The top and front view of an aero-plane are shown in Fig .
- *Let engine or propeller* rotates in the clockwise direction when seen from the rear or tail end and the aero-plane takes a turn to the left.







ω = Angular velocity of the engine in rad/s,

m = Mass of the engine and the propeller in kg,

k = Its radius of gyration in metres,

I = Mass moment of inertia of the engine and the propeller in kg-m^2

$= m.k^2$,

v = Linear velocity of the aeroplane in m/s,

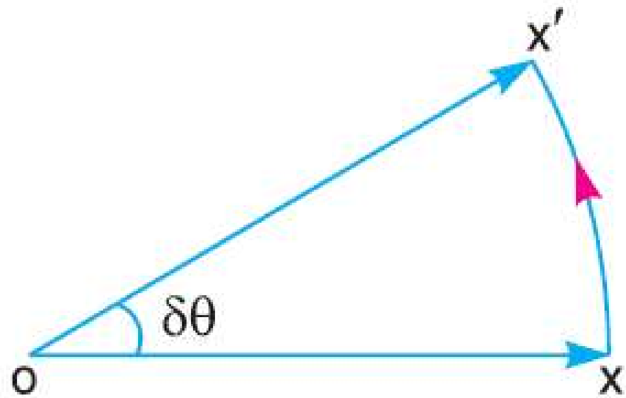
R = Radius of curvature in metres, and

ω_P = Angular velocity of precession $= \dfrac{v}{R}$ rad/s

∴ Gyroscopic couple acting on the aeroplane,

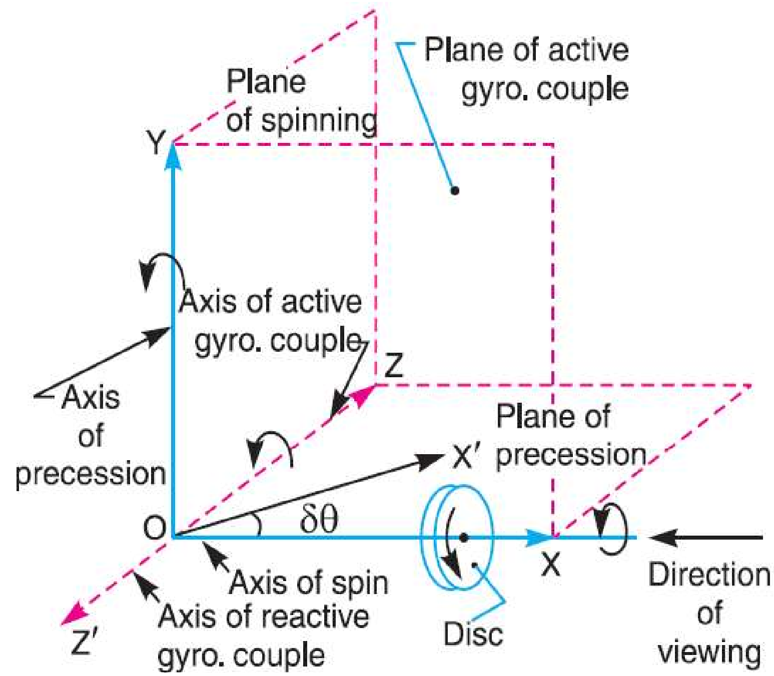
$$C = I.\omega.\omega_P$$

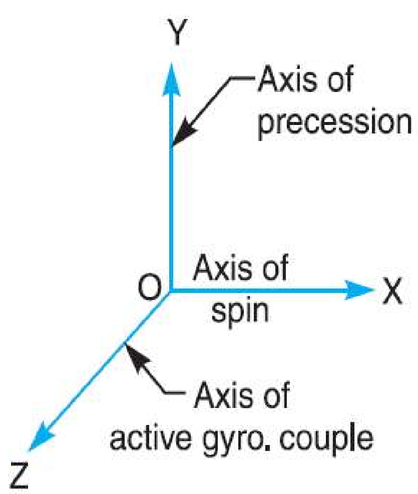
- Before taking the left turn, the angular momentum vector is represented by *ox.*
- *When it takes* left turn, the active gyroscopic couple will change the direction of the angular momentum vector from *ox to ox' as shown in Fig (a).*
- *The vector xx', in the limit, represents the change of angular* momentum or the active gyroscopic couple and is perpendicular to *ox.*
- *Thus the plane of active* gyroscopic couple *XOY will be perpendicular to xx' , i.e. vertical in this case, as shown in Fig* (*b*)*.*

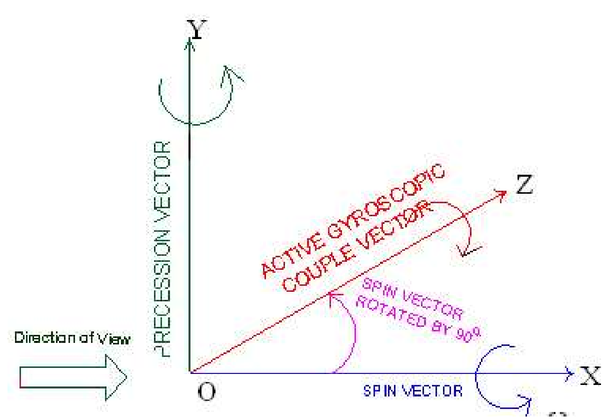


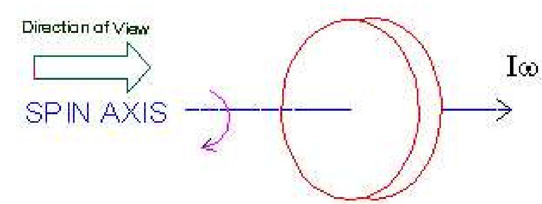
(*a*) Aeroplane taking left turn.

active gyroscopic couple

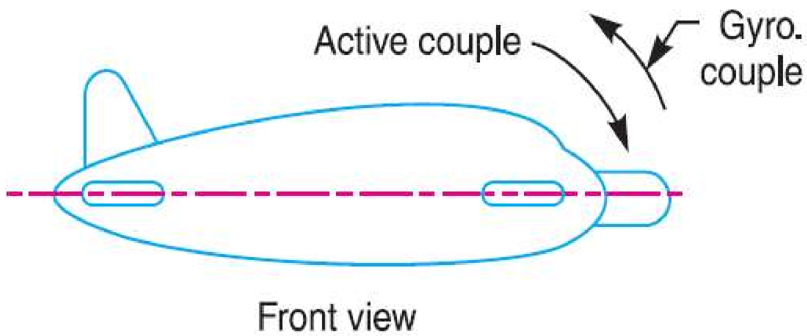


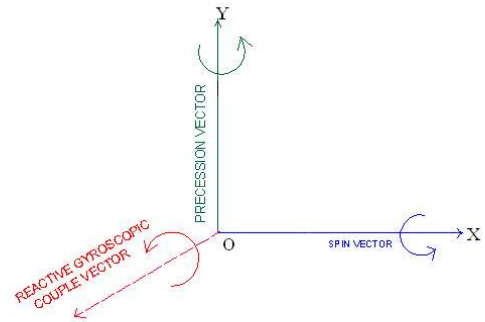


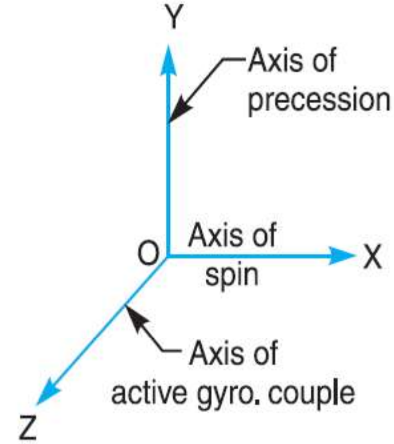




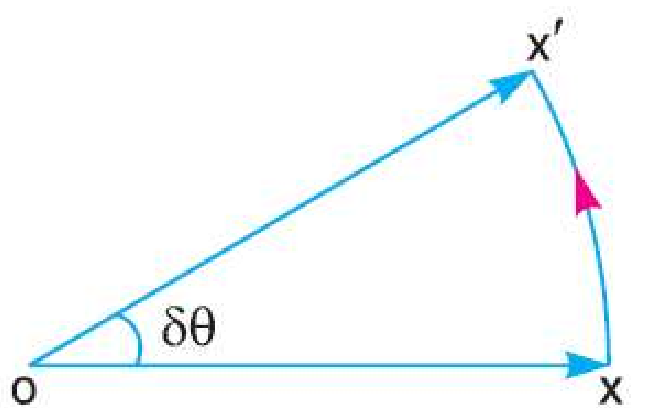
- *By applying right hand screw rule to vector xx', we find that the direction of active gyroscopic* couple is clockwise as shown in the front view of Fig.



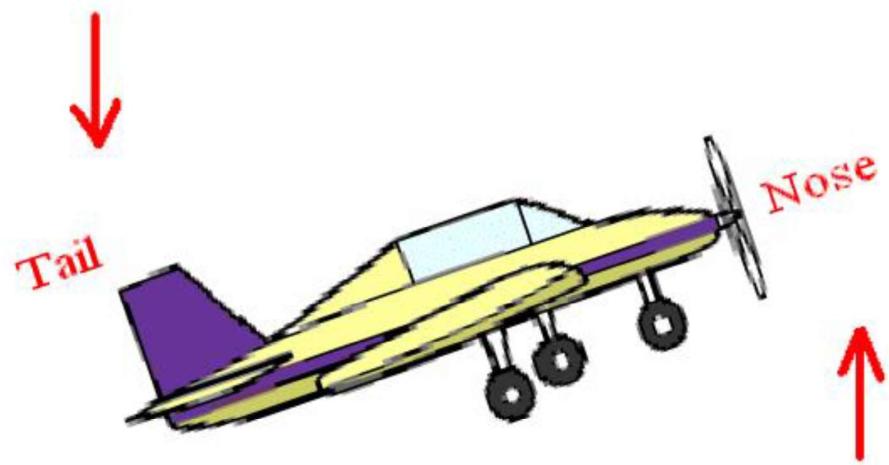




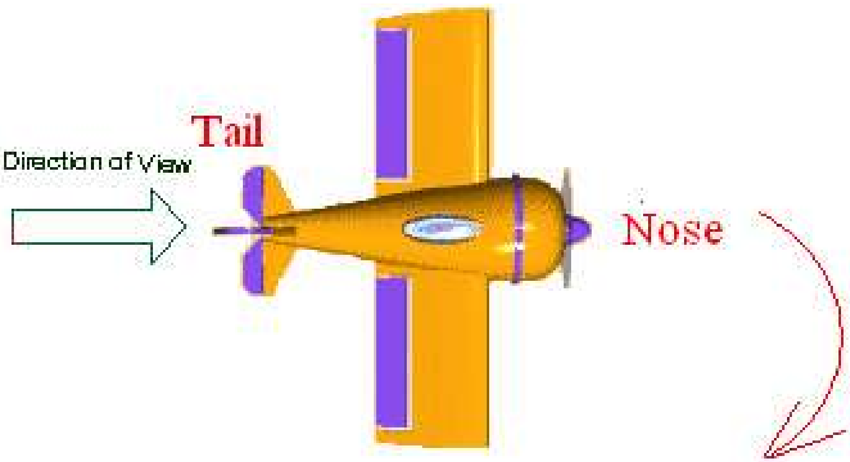
- *In other words, for left hand turning,* the active gyroscopic couple on the aeroplane in the axis *OZ will be clockwise as shown in Fig.*
- *The reactive gyroscopic couple (equal in magnitude of active gyroscopic couple) will act in the* opposite direction (*i.e. in the anticlockwise direction) and the effect of this couple is, therefore, to* raise the nose and dip the tail of the aeroplane.

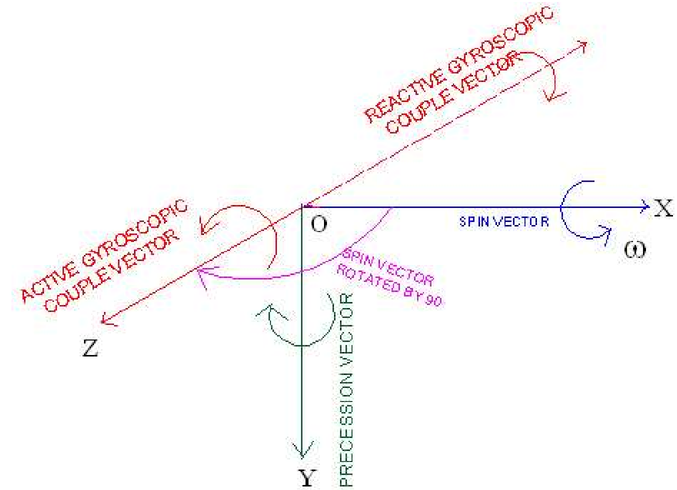


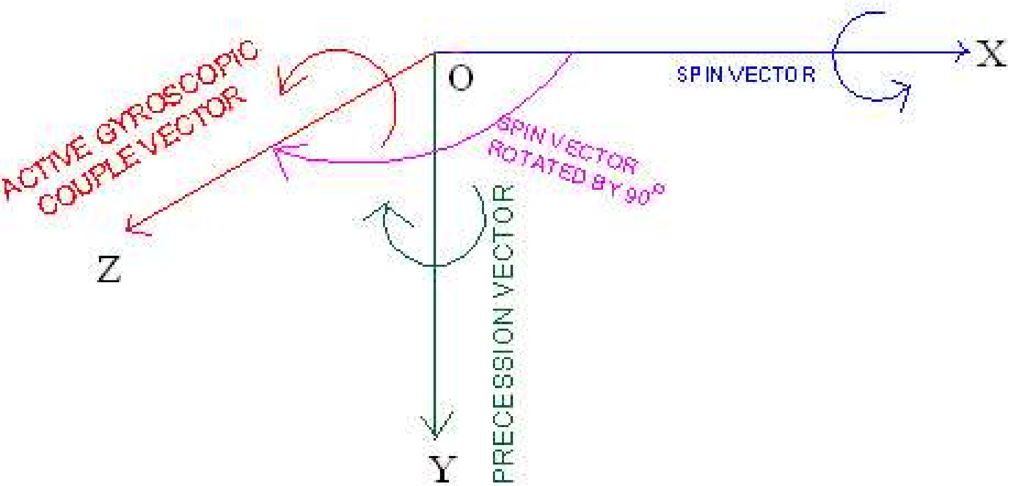
(*a*) Aeroplane taking left turn.



Case (ii): PROPELLER rotates in CLOCKWISE direction when seen from rear end and Aeroplane turns towards RIGHT

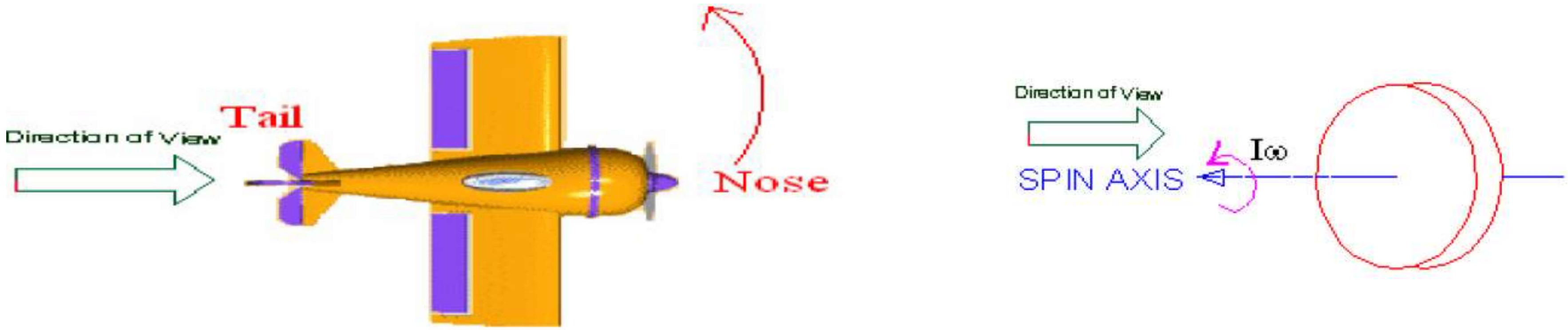


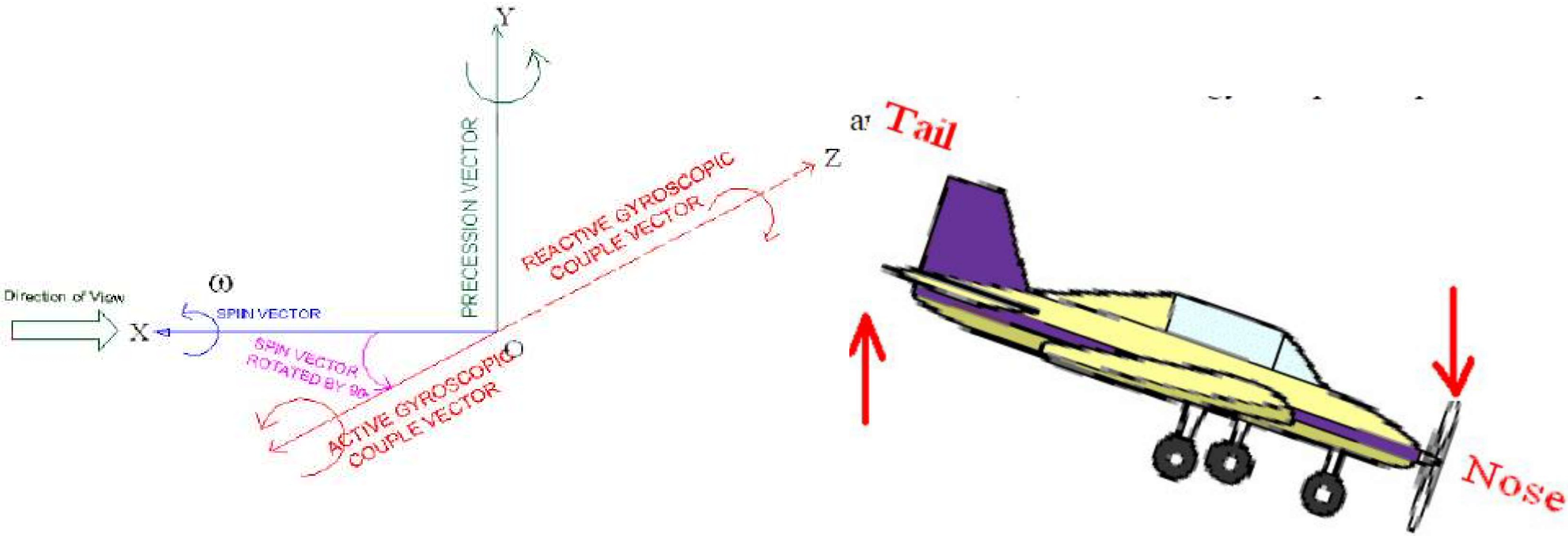




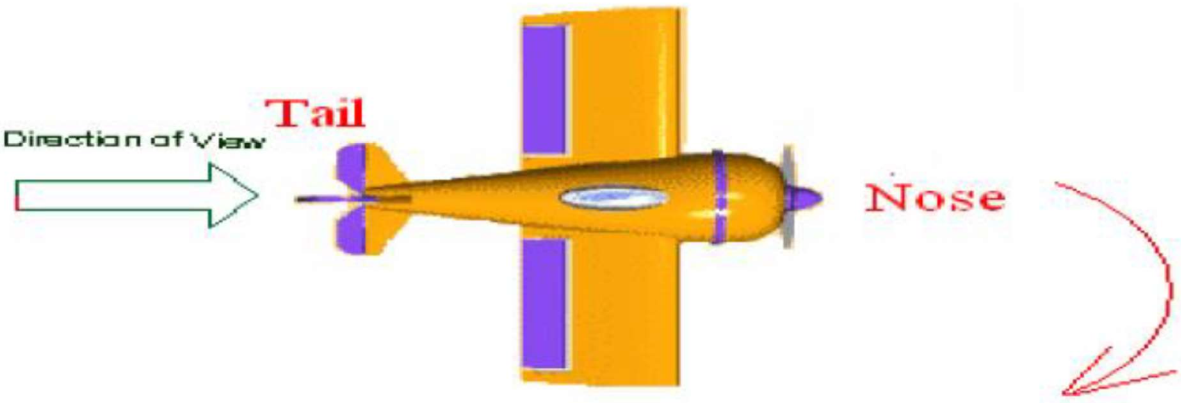
Case (iii):

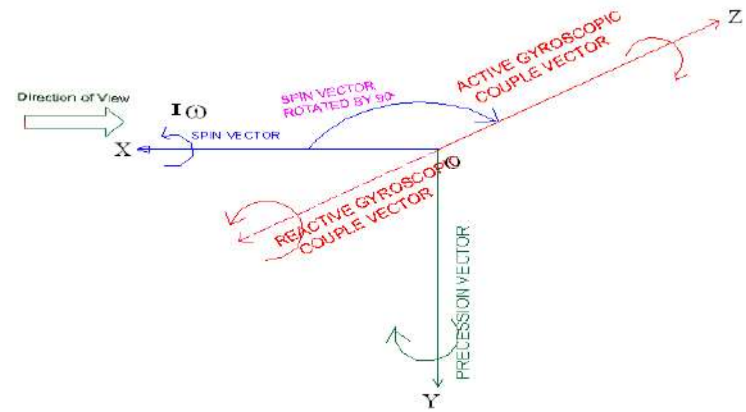
PROPELLER rotates in **ANTICLOCKWISE** direction when seen from rear end and Aeroplane turns towards **LEFT**

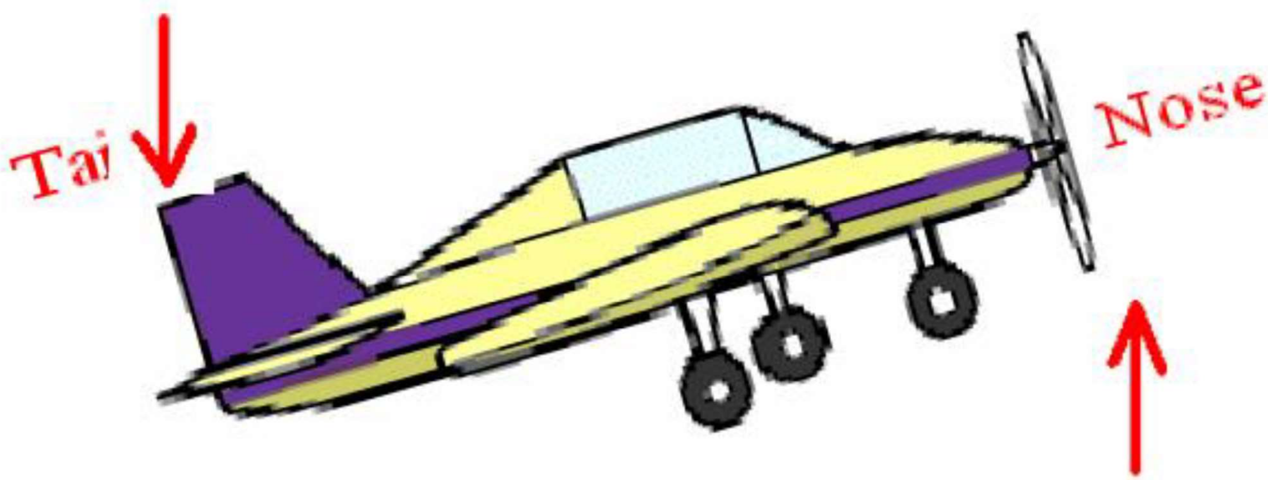




Case (iv): PROPELLER rotates in ANTICLOCKWISE direction when seen from rear end and Aeroplane turns towards RIGHT







Notes :

1. When the aeroplane takes a right turn under similar conditions as discussed above, the effect of the reactive gyroscopic couple will be to dip the nose and raise the tail of the aeroplane.
2. When the engine or propeller rotates in anticlockwise direction when viewed from the rear or tail end and the aeroplane takes a left turn, then the effect of reactive gyroscopic couple will be to dip the nose and raise the tail of the aeroplane.
3. When the aeroplane takes a right turn under similar conditions as mentioned in note 2 above, the effect of reactive gyroscopic couple will be to raise the nose and dip the tail of the aeroplane.
4. When the engine or propeller rotates in clockwise direction when viewed from the front and the aeroplane takes a left turn, then the effect of reactive gyroscopic couple will be to raise the tail and dip the nose of the aeroplane.
5. When the aeroplane takes a right turn under similar conditions as mentioned in note 4-above, the effect of reactive gyroscopic couple will be to raise the nose and dip the tail of the aeroplane.

Example 14.3. *An aeroplane makes a complete half circle of 50 metres radius, towards left, when flying at 200 km per hr. The rotary engine and the propeller of the plane has a mass of 400 kg and a radius of gyration of 0.3 m. The engine rotates at 2400 r.p.m. clockwise when viewed from the rear. Find the gyroscopic couple on the aircraft and state its effect on it.*

Solution. Given : R = 50 m ; v = 200 km/hr = 55.6 m/s ; m = 400 kg ; k = 0.3 m ; N = 2400 r.p.m. or $\omega = 2\pi \times 2400/60 = 251$ rad/s

We know that mass moment of inertia of the engine and the propeller,

$$I = m.k^2 = 400(0.3)^2 = 36 \text{ kg-m}^2$$

and angular velocity of precession,

$$\omega_P = v/R = 55.6/50 = 1.11 \text{ rad/s}$$

We know that gyroscopic couple acting on the aircraft,

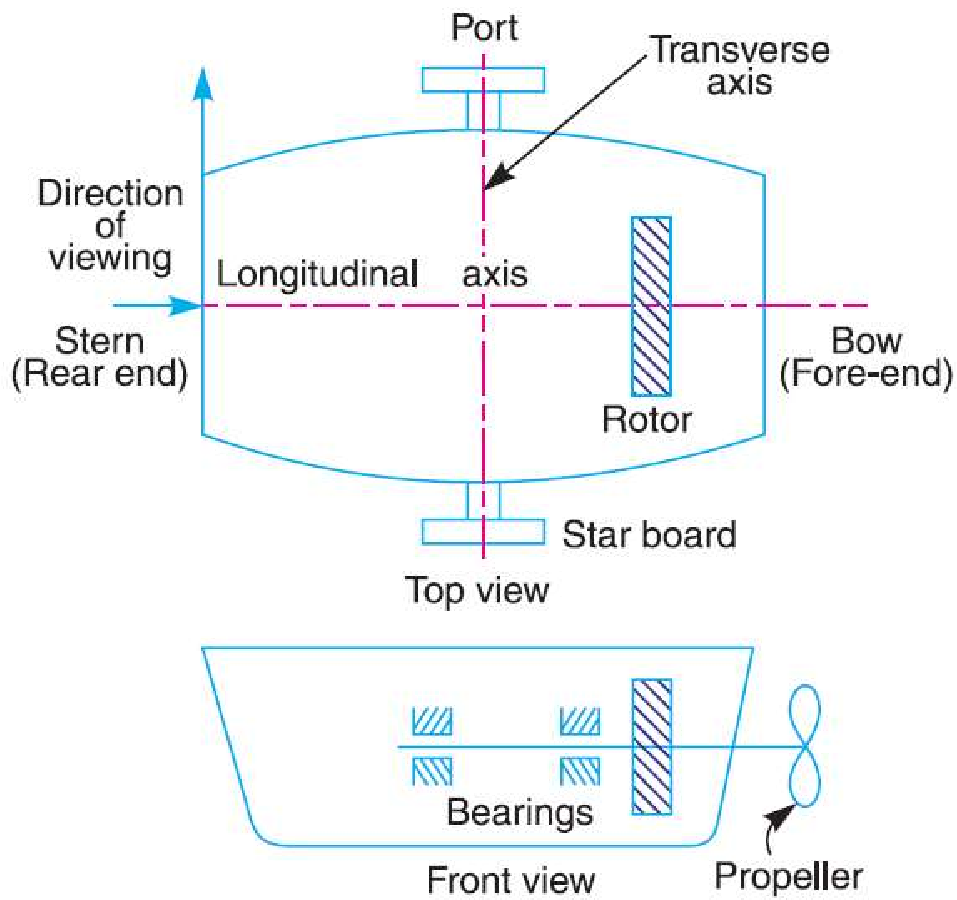
$$C = I.\ \omega.\ \omega_P = 36 \times 251.4 \times 1.11 = 100\ 46 \text{ N-m}$$

$$= 10.046 \text{ kN-m}$$ **Ans.**

when the aero-plane turns towards left, the effect of the gyroscopic couple is to lift the nose upwards and tail downwards.

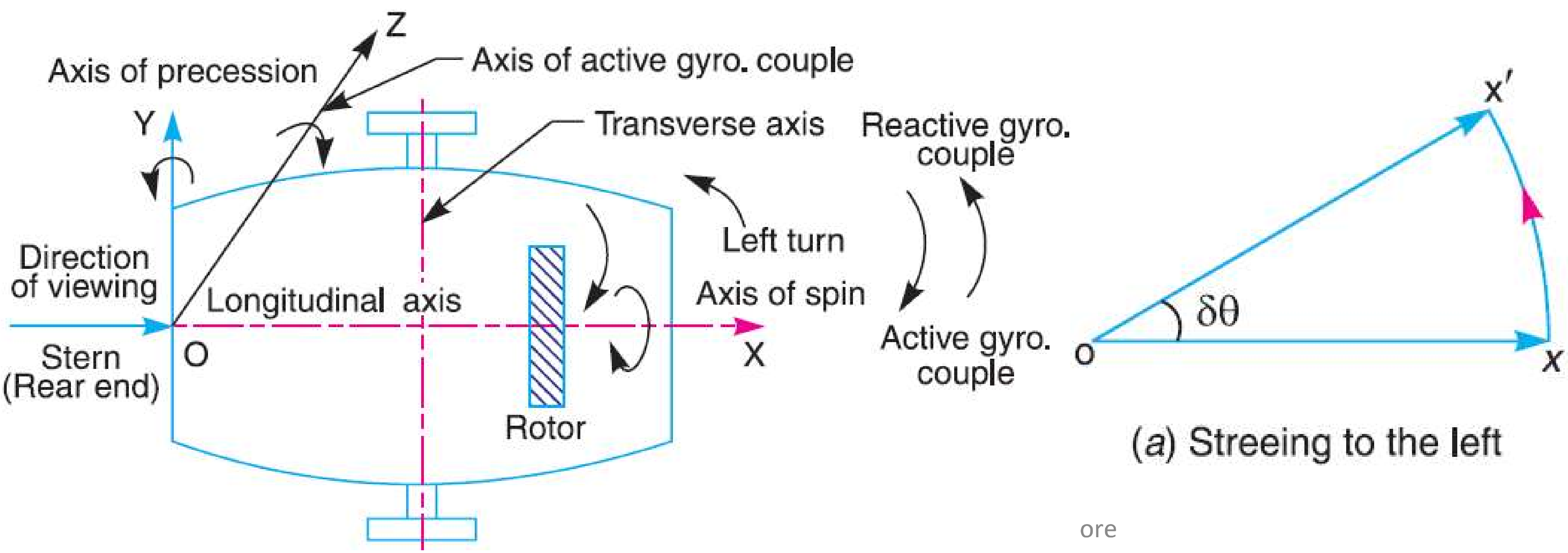
Terms Used in a Naval Ship

- The top and front views of a naval ship are shown in Fig . The fore end of the ship is called *bow and the rear end is known as stern or aft. The left hand and right hand sides of the ship,* when viewed from the stern are called *port and star-board respectively.*
- *We shall now discuss the* effect of gyroscopic couple on the naval ship in the following three cases:
  1. Steering
  2. Pitching, and
  3. Rolling



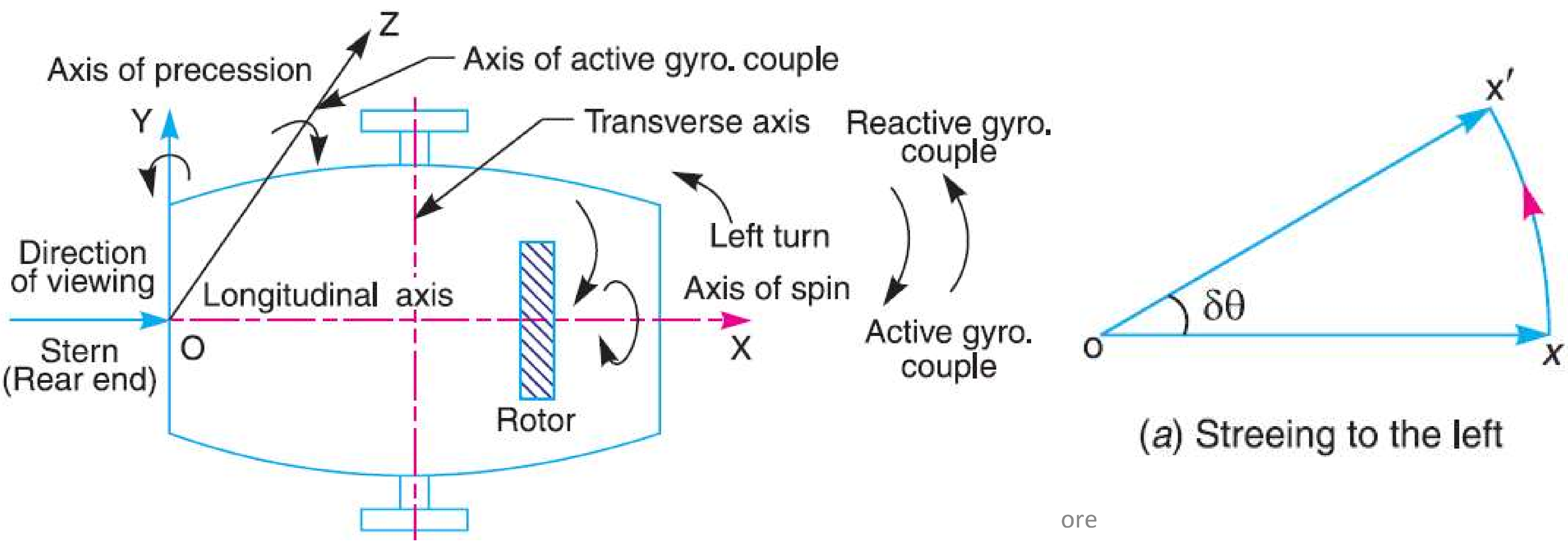
Effect of Gyroscopic Couple on a Naval Ship during Steering

- Steering is the turning of a complete ship in a curve towards left or right, while it moves forward.
- Consider the ship taking a left turn, and rotor rotates in the clockwise direction when viewed from the stern, as shown in Fig.
- The effect of gyroscopic couple on a naval ship during steering taking left or right turn may be obtained in the similar way as for an aeroplane.



(a) Streeing to the left

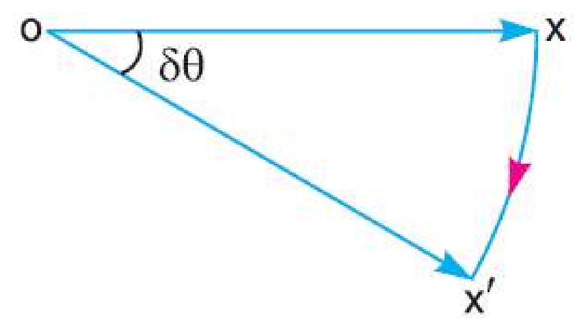
- When the rotor of the ship rotates in the clockwise direction when viewed from the stern, it will have its angular momentum vector in the direction *ox as shown in Fig (a).*
- *As the ship steers to the* left, the active gyroscopic couple will change the angular momentum vector from *ox to ox'.*
- *The* vector *xx' now represents the active gyroscopic couple and is perpendicular to ox. Thus the plane of* active gyroscopic couple is perpendicular to *xx' and its direction in the axis OZ for left hand turn is* clockwise as shown in Fig.
- The reactive gyroscopic couple of the same magnitude will act in the opposite direction (*i.e. in anticlockwise direction).*
- *The effect of this reactive gyroscopic couple is to raise the bow and lower the stern.*



(a) Streeing to the left

Notes:

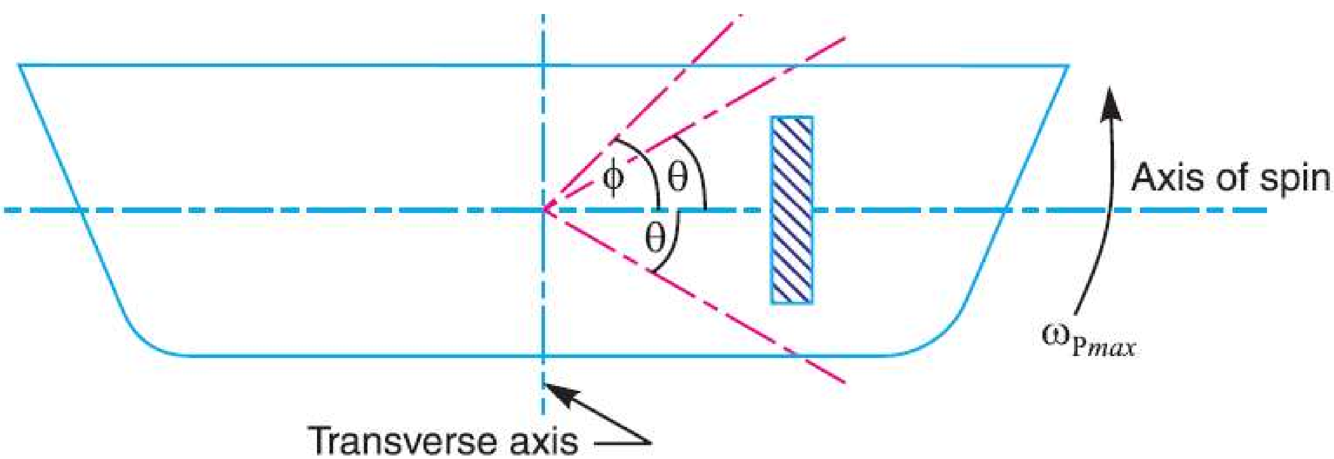
When the ship steers to the right under similar conditions as discussed above, the effect of the reactive gyroscopic couple, as shown in Fig.(*b), will* be to raise the stern and lower the bow.

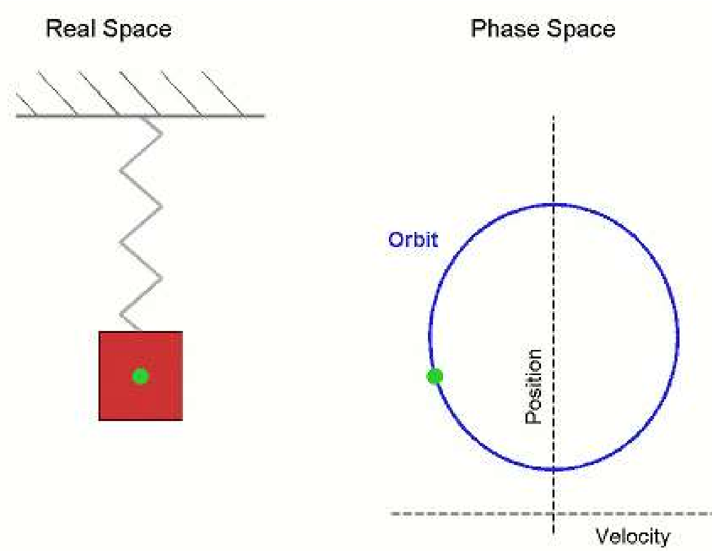


(*b*) Streeing to the right

Effect of Gyroscopic Couple on a Naval Ship during Pitching

- Pitching is the movement of a complete ship up and down in a vertical plane about transverse axis, as shown in Fig.
- *In this case, the transverse axis is the axis of precession. The pitching* of the ship is assumed to take place with simple harmonic motion *i.e. the motion of the axis of spin* about transverse axis is simple harmonic.





$\therefore$ Angular displacement of the axis of spin from mean position after time t seconds,

$$\theta = \phi \sin \omega_1 . t$$

ϕ = Amplitude of swing *i.e.* maximum angle turned from the mean position in radians, and

ω_1 = Angular velocity of S.H.M.

$$= \frac{2\pi}{\text{Time period of S.H.M. in seconds}} = \frac{2\pi}{t_p} \text{ rad/s}$$

Angular velocity of precession,

$$\omega_P = \frac{d\theta}{dt} = \frac{d}{dt}(\phi \sin \omega_1 . t) = \phi\, \omega_1 \cos \omega_1 t$$

The angular velocity of precession will be maximum, if $\cos \omega_1 . t = 1$.

$\therefore$ Maximum angular velocity of precession,

$$\omega_{Pmax} = \phi . \omega_1 = \phi \times 2\pi / t_p \qquad \text{...(Substituting } \cos \omega_1 . t = 1\text{)}$$

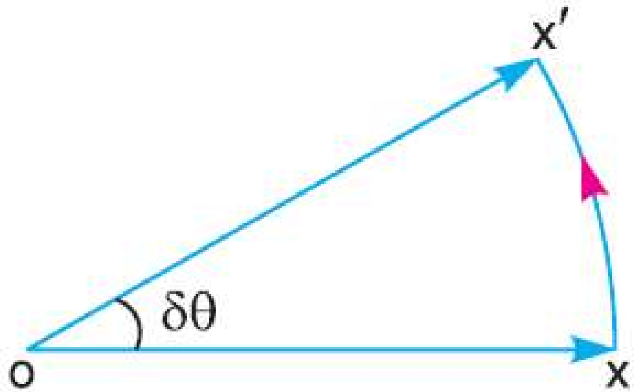
Let I = Moment of inertia of the rotor in kg-m^2, and

ω = Angular velocity of the rotor in rad/s.

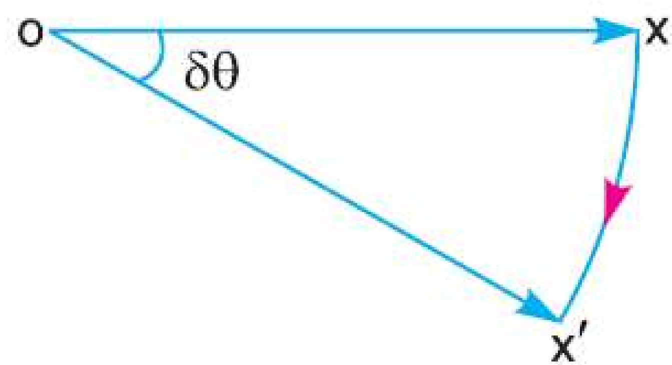
$\therefore$ Mamimum gyroscopic couple,

$$C_{max} = I .\ \omega .\ \omega_{Pmax}$$

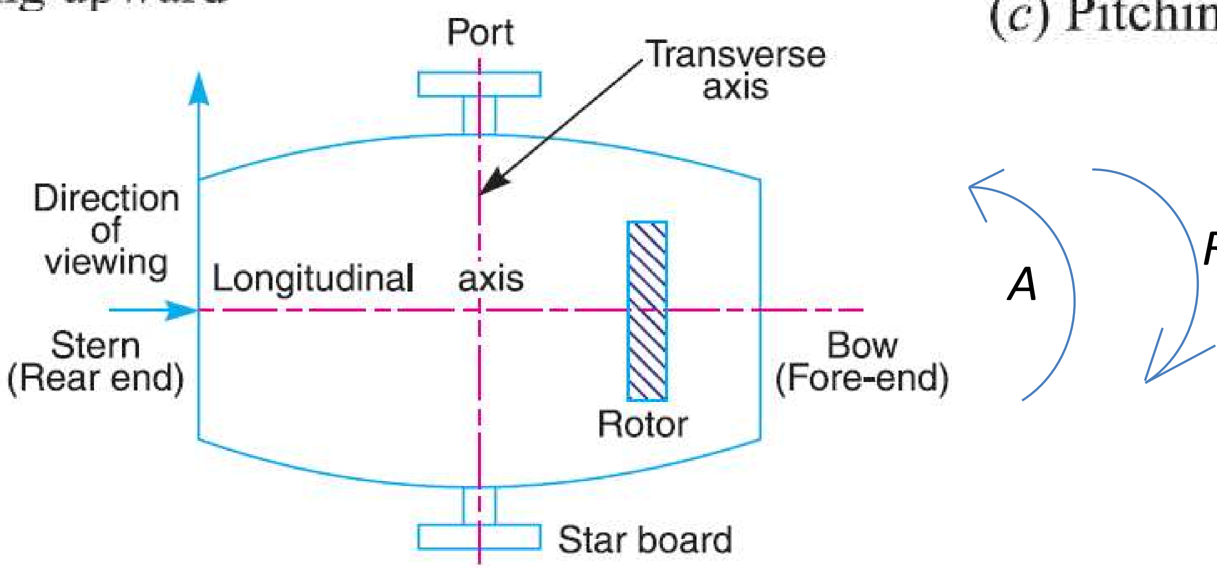
- When the pitching is upward, the effect of the reactive gyroscopic couple, as shown in Fig.(*b), will try to move the ship toward star-board.*
- *On the other hand, if the pitching is downward, the effect* of the reactive gyroscopic couple, as shown in Fig.(*c), is to turn the ship towards port side.*



(*b*) Pitching upward



(*c*) Pitching downward



Notes :

- The effect of the gyroscopic couple is always given on specific position of the axis of spin *i.e.* whether it is pitching downwards or upwards.
- The pitching of a ship produces forces on the bearings which act horizontally and perpendicular to the motion of the ship.
- The maximum gyroscopic couple tends to shear the holding-down bolts.
- The angular acceleration during pitching is given by

$$\alpha = \frac{d^2\theta}{dt^2} = -\phi(\omega_1)^2 \sin \omega_1 t \qquad \ldots \left(\text{Differentiating } \frac{d\theta}{dt} \text{ with respect to } t \right)$$

The angular acceleration is maximum, if $\sin \omega_1 t = 1$.

$\therefore$ Maximum angular acceleration during pitching,

$$\alpha_{max} = (\omega_1)^2$$

Effect of Gyroscopic Couple on a Naval Ship during Rolling

- We know that, for the effect of gyroscopic couple to occur, the axis of precession should always be perpendicular to the axis of spin.
- If, however, the axis of precession becomes parallel to the axis of spin, there will be no effect of the gyroscopic couple acting on the body of the ship.
- In case of rolling of a ship, the axis of precession (*i.e. longitudinal axis) is always parallel to* the axis of spin for all positions.
- Hence, there is no effect of the gyroscopic couple acting on the body of a ship.

Example 14.4. *The turbine rotor of a ship has a mass of 8 tonnes and a radius of gyration 0.6 m. It rotates at 1800 r.p.m. clockwise, when looking from the stern. Determine the gyroscopic couple, if the ship travels at 100 km/hr and steer to the left in a curve of 75 m radius.*

Solution. Given: $m = 8$ t $= 8000$ kg ; $k = 0.6$ m ; $N = 1800$ r.p.m. or $\omega = 2\pi \times 1800/60 = 188.5$ rad/s ; $v = 100$ km/h $= 27.8$ m/s ; $R = 75$ m

We know that mass moment of inertia of the rotor,

$$I = m.k^2 = 8000\,(0.6)^2 = 2880 \text{ kg-m}^2$$

and angular velocity of precession,

$$\omega_P = v / R = 27.8 / 75 = 0.37 \text{ rad/s}$$

We know that gyroscopic couple,

$$C = I.\omega.\omega_P = 2880 \times 188.5 \times 0.37 = 200\,866 \text{ N-m}$$

$$= 200.866 \text{ kN-m} \quad \textbf{Ans.}$$

when the rotor rotates in clockwise direction when looking from the stern and the ship steers to the left, the effect of the reactive gyroscopic couple is to raise the bow and lower the stern.

- *The heavy turbine rotor of a sea vessel rotates at 1500 r.p.m. clockwise looking from the stern, its mass being 750 kg. The vessel pitches with an angular velocity of 1 rad/s. Determine the gyroscopic couple transmitted , when bow is rising, if the radius of gyration for the rotor is 250 mm. Also show in what direction the couple acts ?*
- Given: *N = 1500 r.p.m. or* = 2 × 1500/60 = 157.1 rad/s; *m = 750 kg;* P = 1 rad/s; *k = 250 mm = 0.25 m*

We know that mass moment of inertia of the rotor,

$$I = m.k^2 = 750\,(0.25)^2 = 46.875 \text{ kg-m}^2$$

$\therefore$ Gyroscopic couple transmitted to the hull (*i.e.* body of the sea vessel),

$$C = I.\omega.\omega_P = 46.875 \times 157.1 \times 1 = 7364 \text{ N-m} = 7.364 \text{ kN-m}$$

when the bow is rising *i.e. when the pitching is upward,* the reactive gyroscopic couple acts in the clockwise direction which moves the sea vessel towards star-board.

Example 14.6. *The turbine rotor of a ship has a mass of 3500 kg. It has a radius of gyration of 0.45 m and a speed of 3000 r.p.m. clockwise when looking from stern. Determine the gyroscopic couple and its effect upon the ship:*

1. when the ship is steering to the left on a curve of 100 m radius at a speed of 36 km/h.

2. when the ship is pitching in a simple harmonic motion, the bow falling with its maximum velocity. The period of pitching is 40 seconds and the total angular displacement between the two extreme positions of pitching is 12 degrees.

Solution. Given : $m = 3500$ kg ; $k = 0.45$ m; $N = 3000$ r.p.m. or $\omega = 2\pi \times 3000/60 = 314.2$ rad/s

1. *When the ship is steering to the left*

Given: $R = 100$ m ; $v =$ km/h = 10 m/s

We know that mass moment of inertia of the rotor,

$$I = m.k^2 = 3500\ (0.45)^2 = 708.75 \text{ kg-m}^2$$

and angular velocity of precession,

$$\omega_P = v/R = 10/100 = 0.1 \text{ rad/s}$$

$\therefore$ Gyroscopic couple,

$$C = I.\omega.\omega_P = 708.75 \times 314.2 \times 0.1 = 22\ 270 \text{ N-m}$$

$$= 22.27 \text{ kN-m}$$ **Ans.**

2. *When the ship is pitching with the bow falling*

Given: $t_p = 40$ s

Since the total angular displacement between the two extreme positions of pitching is 12° (*i.e.* $2\phi = 12°$), therefore amplitude of swing,

$$\phi = 12 / 2 = 6° = 6 \times \pi/180 = 0.105 \text{ rad}$$

and angular velocity of the simple harmonic motion,

$$\omega_1 = 2\pi / t_p = 2\pi / 40 = 0.157 \text{ rad/s}$$

We know that maximum angular velocity of precession,

$$\omega_P = \phi.\omega_1 = 0.105 \times 0.157 = 0.0165 \text{ rad/s}$$

$\therefore$ Gyroscopic couple,

$$C = I.\omega.\omega_P = 708.75 \times 314.2 \times 0.0165 = 3675 \text{ N-m}$$

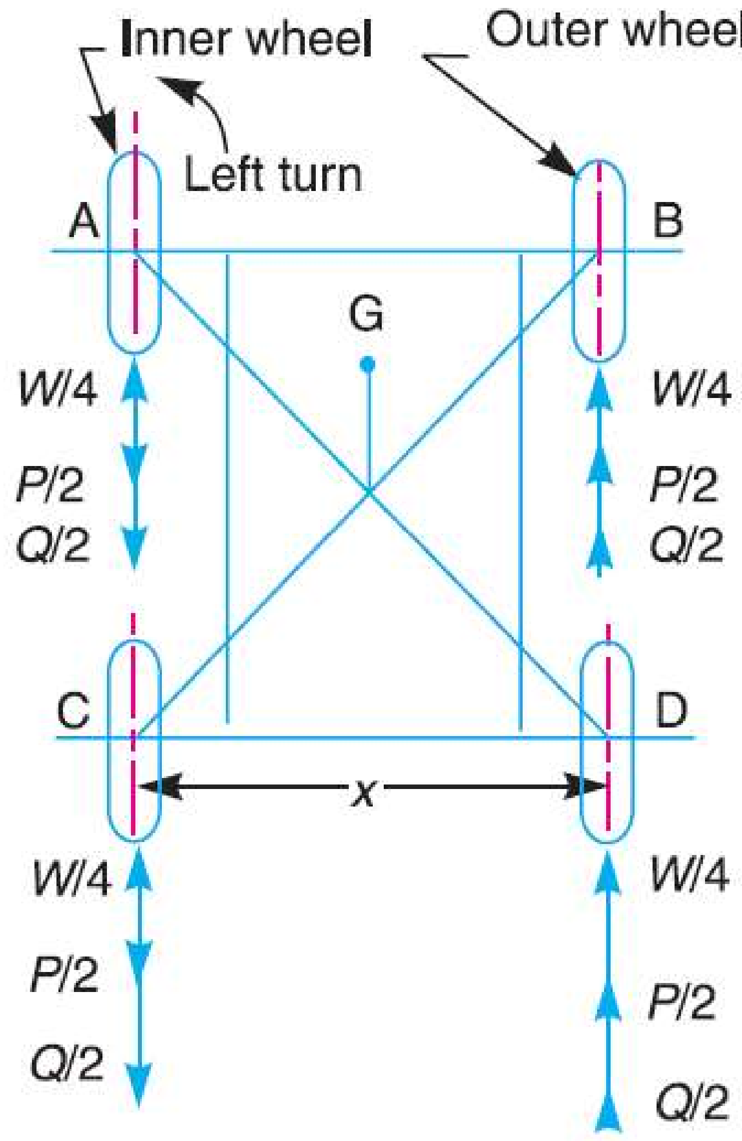
$$= 3.675 \text{ kN-m}$$ **Ans.**

when the bow is falling (*i.e. when the pitching is downward),* the effect of the reactive gyroscopic couple is to move the ship towards port side.

Stability of a Four Wheel Drive Moving in a Curved Path

- Consider the four wheels *A, B, C and D of an* automobile locomotive taking a turn towards left as shown in Fig.

The wheels *A and C are inner wheels, whereas B and D are outer wheels. The centre of gravity (C.G.) of* the vehicle lies vertically above the road surface.



Let m = Mass of the vehicle in kg,

W = Weight of the vehicle in newtons = $m.g$,

r_W = Radius of the wheels in metres,

R = Radius of curvature in metres ($R > r_W$),

h = Distance of centre of gravity, vertically above the road surface in metres,

x = Width of track in metres,

I_W = Mass moment of inertia of one of the wheels in kg-m^2,

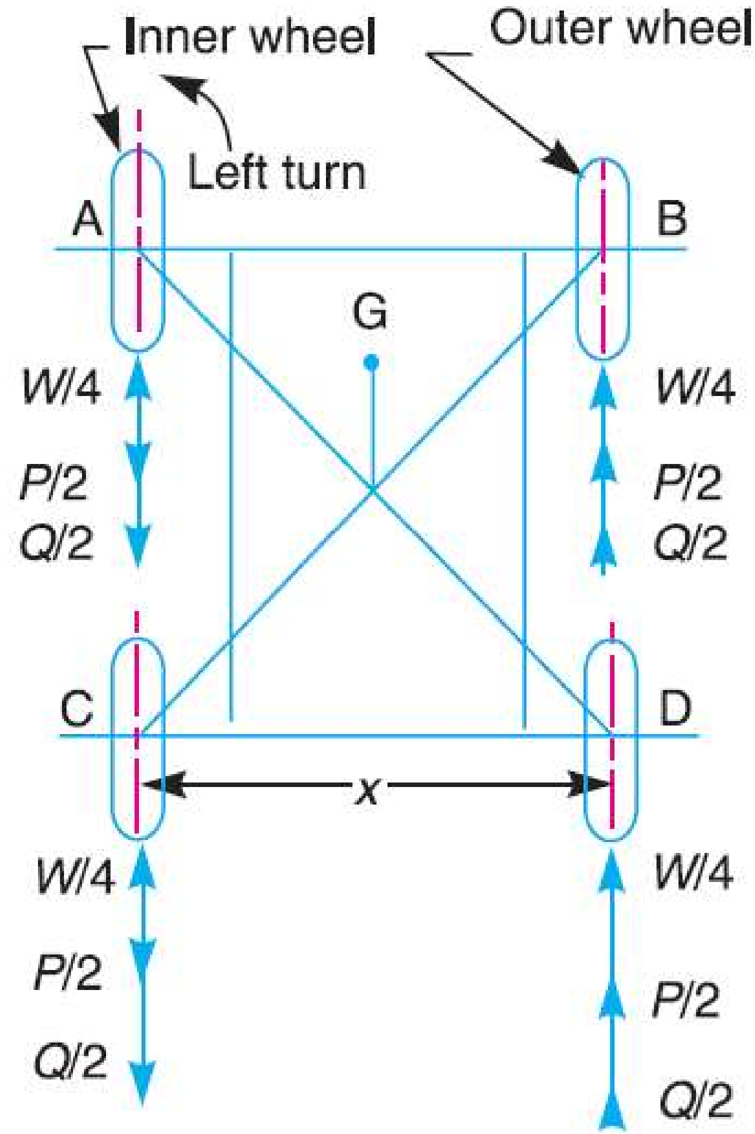
ω_W = Angular velocity of the wheels or velocity of spin in rad/s,

I_E = Mass moment of inertia of the rotating parts of the engine in kg-m^2,

ω_E = Angular velocity of the rotating parts of the engine in rad/s,

G = Gear ratio = ω_E / ω_W,

v = Linear velocity of the vehicle in m/s = $\omega_W . r_W$



A little consideration will show, that the weight of the vehicle (*W) will be* equally distributed over the four wheels which will act downwards.
The reaction between each wheel and the road surface of the same magnitude will act upwards.
Therefore Road reaction over each wheel, = *W/4 = m.g /4 newtons*

- Let us now consider the effect of the gyroscopic couple and centrifugal couple on the vehicle.

1. *Effect of the gyroscopic couple*

Since the vehicle takes a turn towards left due to the precession and other rotating parts, therefore a gyroscopic couple will act.

We know that velocity of precession,

$$\omega_P = v/R$$

$\therefore$ Gyroscopic couple due to 4 wheels,

$$C_W = 4\, I_W.\omega_W.\omega_P$$

and gyroscopic couple due to the rotating parts of the engine,

$$C_E = I_E.\omega_E.\omega_P = I_E.G.\omega_W.\omega_P \qquad \text{... } (\because G = \omega_E/\omega_W)$$

$\therefore$ Net gyroscopic couple,

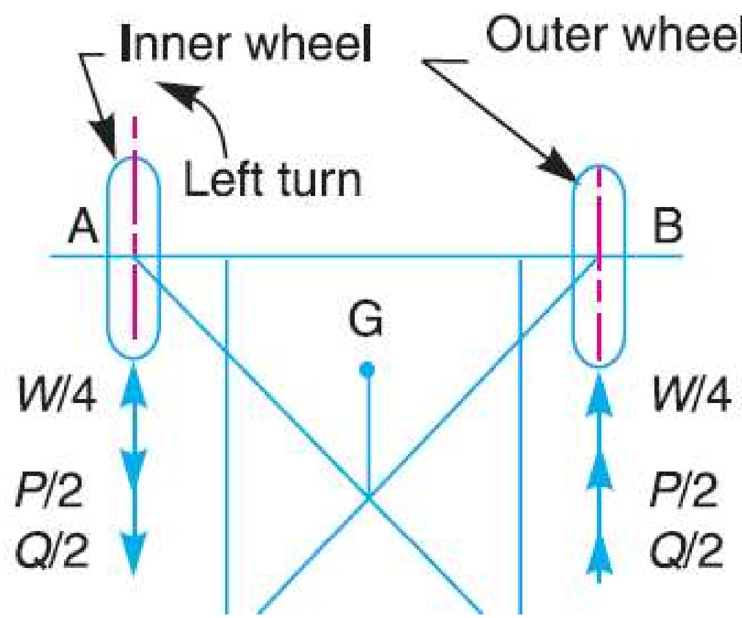
$$C = C_W \pm C_E = 4\, I_W.\omega_W.\omega_P \pm I_E.G.\omega_W.\omega_P$$

$$= \omega_W.\omega_P\,(4\, I_W \pm G.I_E)$$

The ***positive sign is used when the wheels and rotating parts of the engine rotate in the same*** direction. If the rotating parts of the engine revolves in opposite direction, then ***negative sign is used.***

- Due to the gyroscopic couple, vertical reaction on the road surface will be produced.
- The reaction will be vertically upwards on the outer wheels and vertically downwards on the inner wheels.
- Let the magnitude of this reaction at the two outer or inner wheels be *P newtons. Then*
- $P \times x = C$ *or* $P = C/x$
- Vertical reaction at each of the outer or inner wheels, $P/2 = C/2x$

Note: We have discussed above that when rotating parts of the engine rotate in opposite directions, then –ve sign is used, *i.e. net gyroscopic couple,*
$C = C_W - C_E$,
When $C_E > C_W$, then *C will be –ve. Thus the reaction will be vertically downwards on the outer wheels* and vertically upwards on the inner wheels.



2. *Effect of the centrifugal couple*

- Since the vehicle moves along a curved path, therefore centrifugal force will act outwardly at the centre of gravity of the vehicle.
- The effect of this centrifugal force is also to overturn the vehicle.
- We know that centrifugal force,

$$F_C = \frac{m \times v^2}{R}$$

$\therefore$ The couple tending to overturn the vehicle or overturning couple,

$$C_O = F_C \times h = \frac{m.v^2}{R} \times h$$

- This overturning couple is balanced by vertical reactions, which are vertically upwards on the outer wheels and vertically downwards on the inner wheels.

- Let the magnitude of this reaction at the two outer or inner wheels be *Q. Then*

$$Q \times x = C_O \quad \text{or} \quad Q = \frac{C_O}{x} = \frac{m.v^2.h}{R.x}$$

$\therefore$ Vertical reaction at each of the outer or inner wheels

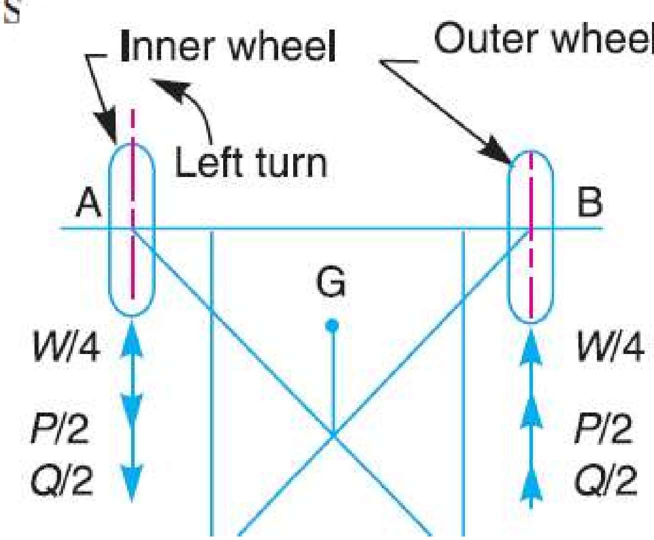
$$\frac{Q}{2} = \frac{m.v^2.h}{2R.x}$$

$\therefore$ Total vertical reaction at each of the outer wheel,

$$P_O = \frac{W}{4} + \frac{P}{2} + \frac{Q}{2}$$

total vertical reaction at each of the inner wheel,

$$P_I = \frac{W}{4} - \frac{P}{2} - \frac{Q}{2}$$



A little consideration will show that when the vehicle is running at high speeds, P_I may be zero or even negative. This will cause the inner wheels to leave the ground thus tending to overturn the automobile. In order to have the contact between the inner wheels and the ground, the sum of *P/2 and Q/2 must be less than W/4.*

A four-wheeled trolley car of mass 2500 kg runs on rails, which are 1.5 m apart and travels around a curve of 30 m radius at 24 km / hr. The rails are at the same level. Each wheel of the trolley is 0.75 m in diameter and each of the two axles is driven by a motor running in a direction opposite to that of the wheels at a speed of five times the speed of rotation of the wheels. The moment of inertia of each axle with gear and wheels is 18 kg-m2. Each motor with shaft and gear pinion has a moment of inertia of 12 kg-m2. The centre of gravity of the car is 0.9 m above the rail level. Determine the vertical force exerted by each wheel on the rails taking into consideration the centrifugal and gyroscopic effects. State the centrifugal and gyroscopic effects on the trolley.

Solution. Given : $m = 2500$ kg ; $x = 1.5$ m ; $R = 30$ m ; $v = 24$ km/h $= 6.67$ m/s ; $d_W = 0.75$ m or $r_W = 0.375$ m ; $G = \omega_E/\omega_W = 5$; $I_W = 18$ kg-m^2 ; $I_E = 12$ kg-m^2 ; $h = 0.9$ m

$\therefore$ Road reaction over each wheel $= W/4 = m.g/4 = 2500 \times 9.81/4 = 6131.25$ N

We know that angular velocity of the wheels,

$$\omega_W = v/r_W = 6.67/0.375 = 17.8 \text{ rad/s}$$

and angular velocity of precession, $\omega_P = v/R = 6.67/30 = 0.22$ rad/s

$\therefore$ Gyroscopic couple due to one pair of wheels and axle,

$$C_W = 2\,I_W.\omega_W.\omega_P = 2 \times 18 \times 17.8 \times 0.22 = 141 \text{ N-m}$$

and gyroscopic couple due to the rotating parts of the motor and gears,

$$C_E = 2\,I_E.\omega_E.\omega_P = 2\,I_E.G.\omega_W.\omega_P \qquad \ldots (\because\ \omega_E = G.\ \omega_W)$$

$$= 2 \times 12 \times 5 \times 17.8 \times 0.22 = 470 \text{ N-m}$$

$\therefore$ Net gyroscopic couple, $C = C_W - C_E = 141 - 470 = -329$ N-m

... (–ve sign is used due to opposite direction of motor)

Due to this net gyroscopic couple, the vertical reaction on the rails will be produced. Since C_E is greater than C_W, therefore the reaction will be vertically downwards on the outer wheels and vertically upwards on the inner wheels. Let the magnitude of this reaction at each of the outer or inner wheel be $P/2$ newton.

$$\therefore \qquad P/2 = C/2x = 329 / 2 \times 1.5 = 109.7 \text{ N}$$

We know that centrifugal force, $F_C = m.v^2/R = 2500\,(6.67)^2/30 = 3707$ N

$\therefore$ Overturning couple, $C_O = F_C \times h = 3707 \times 0.9 = 3336.3$ N-m

This overturning couple is balanced by the vertical reactions which are vertically upwards on the outer wheels and vertically downwards on the inner wheels. Let the magnitude of this reaction at each of the outer or inner wheels be $Q/2$ newton.

$$Q/2 = C_O / 2x = 3336.3 / 2 \times 1.5 = 1112.1 \text{ N}$$

We know that vertical force exerted on each outer wheel,

$$P_O = \frac{W}{4} - \frac{P}{2} + \frac{Q}{2} = 6131.25 - 109.7 + 1112.1 = 7142.65 \text{ N } \textbf{Ans.}$$

and vertical force exerted on each inner wheel,

$$P_I = \frac{W}{4} + \frac{P}{2} - \frac{Q}{2} = 6131.25 + 109.7 - 1112.1 = 5128.85 \text{ N } \textbf{Ans.}$$

A *rear engine automobile is travelling along a track of 100 metres mean radius. Each of the four road wheels has a moment of inertia of 2.5 kg-m^2 and an effective diameter of 0.6 m. The rotating parts of the engine have a moment of inertia of 1.2 kg-m^2. The engine axis is parallel to the rear axle and the crankshaft rotates in the same sense as the road wheels. The ratio of engine speed to back axle speed is 3:1. The automobile has a mass of 1600 kg and has its centre of gravity 0.5 m above road level. The width of the track of the vehicle is 1.5 m.*
Determine the limiting speed of the vehicle around the curve for ail four wheels to maintain contact with the road surface. Assume that the road surface is not cambered and centre of gravity of the automobile lies centrally with respect to the four wheels.

Solution. Given : $R = 100$ m ; $I_W = 2.5$ kg-m^2 ; $d_W = 0.6$ m or $r_W = 0.3$ m ; $I_E = 1.2$ kg-m^2; $G = \omega_E/\omega_W = 3$; $m = 1600$ kg ; $h = 0.5$ m ; $x = 1.5$ m

Road reaction over each wheel

$$= W/4 = m.g / 4 = 1600 \times 9.81/4 = 3924 \text{ N}$$

$$\omega_W = \frac{v}{r_W} = \frac{v}{0.3} = 3.33\ v \text{ rad/s} \qquad \omega_P = \frac{v}{R} = \frac{v}{100} = 0.01\ v \text{ rad/s}$$

$$C_W = 4\, I_W.\omega_W.\omega_P = 4 \times 2.5 \times \frac{v}{0.3} \times \frac{v}{100} = 0.33\ v^2 \text{ N-m}$$

$$C_E = I_E.\omega_E.\omega_P = I_E.G.\omega_W.\omega_P$$
$$= 1.2 \times 3 \times 3.33v \times 0.01v = 0.12\ v^2 \text{ N-m}$$

Total gyroscopic couple,

$$C = C_W + C_E = 0.33\ v^2 + 0.12\ v^2 = 0.45\ v^2 \text{ N-m}$$

$$P/2 = C/2x = 0.45v^2/2 \times 1.5 = 0.15\ v^2 \text{ N}$$

centrifugal force,

$$F_C = m.v^2/R = 1600 \times v^2/100 = 16\ v^2 \text{ N}$$

$$C_O = F_C \times h = 16\ v^2 \times 0.5 = 8\ v^2 \text{ N-m}$$

$$Q\ /\ 2 = C_O\ /\ 2x = 8\ v^2/\ 2 \times 1.5 = 2.67\ v^2 \text{ N}$$

$$P_O = \frac{W}{4} + \frac{P}{2} + \frac{Q}{2}$$

$$P_I = \frac{W}{4} - \frac{P}{2} - \frac{Q}{2} = \frac{W}{4} - \left(\frac{P}{2} + \frac{Q}{2}\right) \qquad \frac{P}{2} + \frac{Q}{2} \le \frac{W}{4}$$

$$0.15\ v^2 + 2.67\ v^2 \le 3924 \quad \text{or} \quad 2.82\ v^2 \le 3924$$

$$v^2 \le 3924/2.82 = 1391.5$$

$$v \le 37.3 \text{ m/s} = 37.3 \times 3600\ /\ 1000 = 134.28 \text{ km/h}$$ **Ans.**

A four wheeled motor car of mass 2000 kg has a wheel base 2.5 m, track width 1.5 m and height of centre of gravity 500 mm above the ground level and lies at 1 metre from the front axle. Each wheel has an effective diameter of 0.8 m and a moment of inertia of0.8kg-m². The drive shaft, engine flywheel and transmission are rotating at 4 times the speed of road wheel, in a clockwise direction when viewed from the front, and is equivalent to a mass of 75 kg having a radius of gyration of 100 mm. If the car is taking a right turn of 60 m radius at 60 km/h, find the load on each wheel.

Solution. Given : $m = 2000$ kg : $b = 2.5$ m ; $x = 1.5$ m ; $h = 500$ mm $= 0.5$ m ; $L = 1$ m ; $d_W =$ 0.8 m or $r_W = 0.4$ m ; $I_W = 0.8$ kg-m² ; $G = \omega_E / \omega_W = 4$; $m_E = 75$ kg ; $k_E = 100$ mm $= 0.1$ m ; $R = 60$ m ; $v = 60$ km/h $= 16.67$ m/s

Since the centre of gravity of the car lies at 1 m from the front axle and the weight of the car *(W = m.g)* lies at the centre of gravity, therefore weight on the front wheels and rear wheels will be different.

Let W_1 = Weight on the front wheels, and

W_2 = Weight on the rear wheels.

Taking moment about the front wheels,

$$W_2 \times 2.5 = W \times 1 = m.g \times 1 = 2000 \times 9.81 \times 1 = 19\,620$$

$$\therefore \quad W_2 = 19\,620 / 2.5 = 7848 \text{ N}$$

We know that weight of the car or on the four wheels,

$$W = W_1 + W_2 = m.g = 2000 \times 9.81 = 19\,620 \text{ N}$$

$$W_1 = W - W_2 = 19\,620 - 7848 = 11\,772 \text{ N}$$

$\therefore$ Weight on each of the front wheels

$$= W_1 / 2 = 11\,772 / 2 = 5886 \text{ N}$$

--

weight on each of the rear wheels

$$= W_2/2 = 7874 / 2 = 3924 \text{ N}$$

We know angular velocity of wheels,

$$\omega_W = v/r_W = 16.67 / 0.4 = 41.675 \text{ rad/s}$$

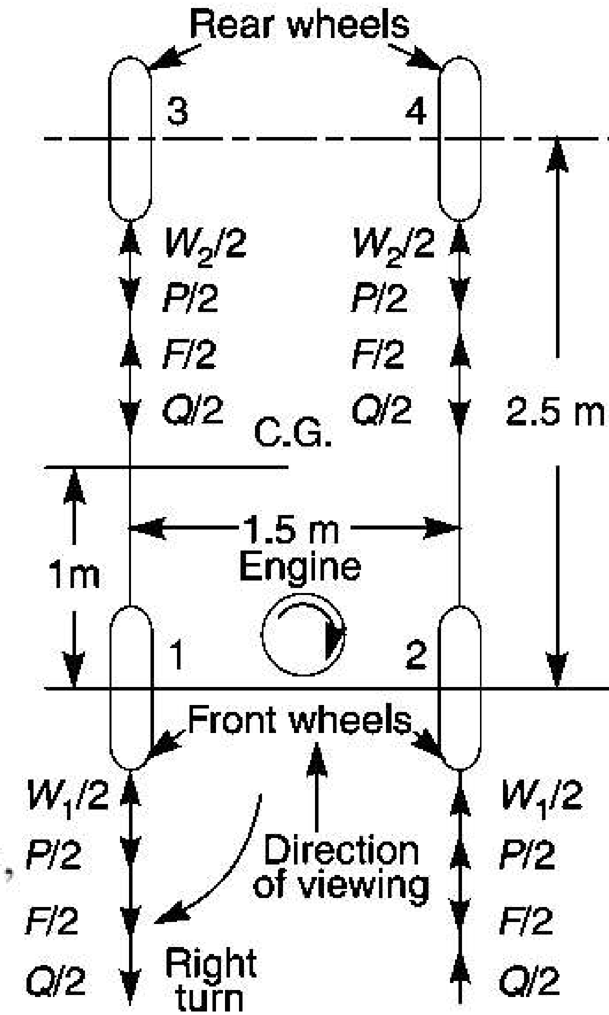
l angular velocity of precession,

$$\omega_P = v/R = 16.67 / 60 = 0.278 \text{ rad/s}$$

Gyroscopic couple due to four wheels,

$$C_W = 4 I_W.\omega_W.\omega_P$$

$$= 4 \times 0.8 \times 41.675 \times 0.278 = 37.1 \text{ N-m}$$



$$\therefore \quad P/2 = C_W / 2x = 37.1 / 2 \times 1.5 = 12.37 \text{ N}$$

We know that mass moment of inertia of rotating parts of the engine,

$$I_E = m_E (k_E)^2 = 75 (0.1)^2 = 0.75 \text{ kg-m}^2$$

$\therefore$ Gyroscopic couple due to rotating parts of the engine,

$$C_E = I_E.\omega_E.\omega_P = m_E (k_E)^2 G.\ \omega_W.\omega_P$$

$$= 75 (0.1)^2 \, 4 \times 41.675 \times 0.278 = 34.7 \text{ N-m}$$

Stability of a Two Wheel Vehicle Taking a Turn

- Consider a two wheel vehicle (say a scooter or motor cycle) taking a right turn as shown in Fig.(a).

m = Mass of the vehicle and its rider in kg,

W = Weight of the vehicle and its rider in newtons = m.g,

h = Height of the centre of gravity of the vehicle and rider,

r_w = Radius of the wheels,

R = Radius of track or curvature,

I_W = Mass moment of inertia of each wheel,

I_E = Mass moment of inertia of the rotating parts of the

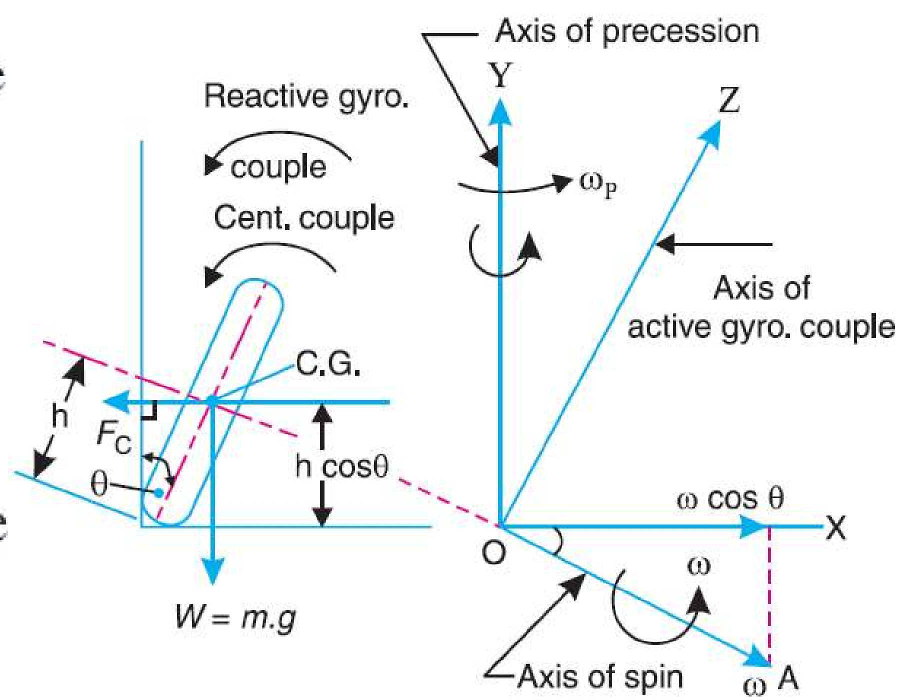
ω_W = Angular velocity of the wheels,

ω_E = Angular velocity of the engine,

G = Gear ratio = ω_E / ω_W,

v = Linear velocity of the vehicle = $\omega_W \times r_W$,

θ = Angle of heel. It is inclination of the vehicle to the vertical for equilibrium.



1. *Effect of gyroscopic couple*

We know that $v = \omega_W \times r_W$ or $\omega_W = v / r_W$

and $$\omega_E = G.\omega_W = G \times \frac{v}{r_W}$$

$\therefore$ Total $$(I \times \omega) = 2\, I_W \times \omega_W \pm I_E \times \omega_E$$

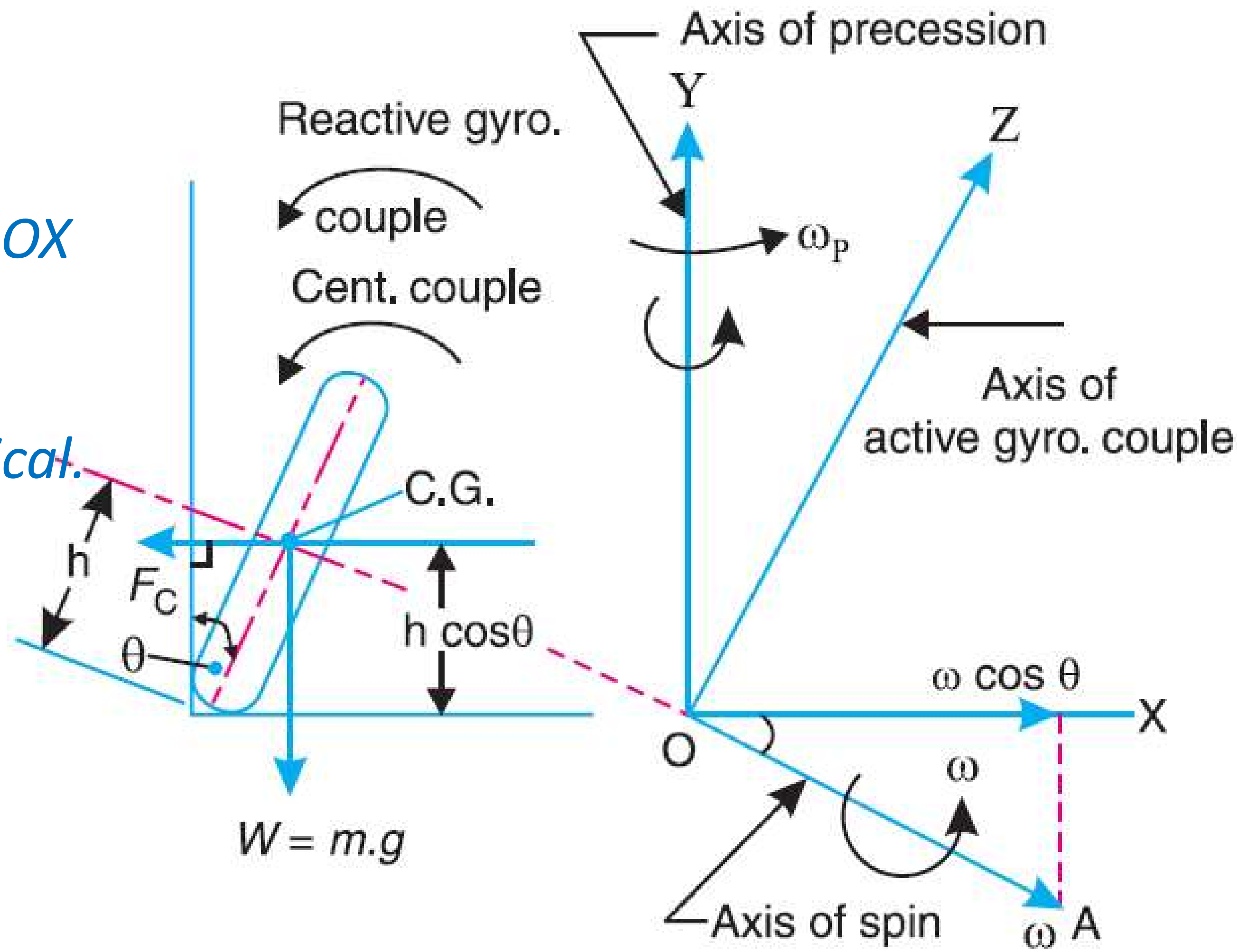
$$= 2\, I_W \times \frac{v}{r_W} \pm I_E \times G \times \frac{v}{r_W} = \frac{v}{r_W}\,(2\, I_W \pm G.I_E)$$

and velocity of precession, $\omega_P = v / R$

- A little consideration will show that when the wheels move over the curved path, the vehicle is always inclined at an angle θ with the vertical plane as shown in Fig.
- This angle is known as *angle of heel (θ).*
- *In other words, the axis of spin is inclined to the horizontal at an angle θ, as shown in Fig.*

Thus the angular momentum vector $I\omega$ due to spin is represented by OA inclined to OX at an angle θ.

But the precession axis is vertical. Therefore the spin vector is resolved along OX.



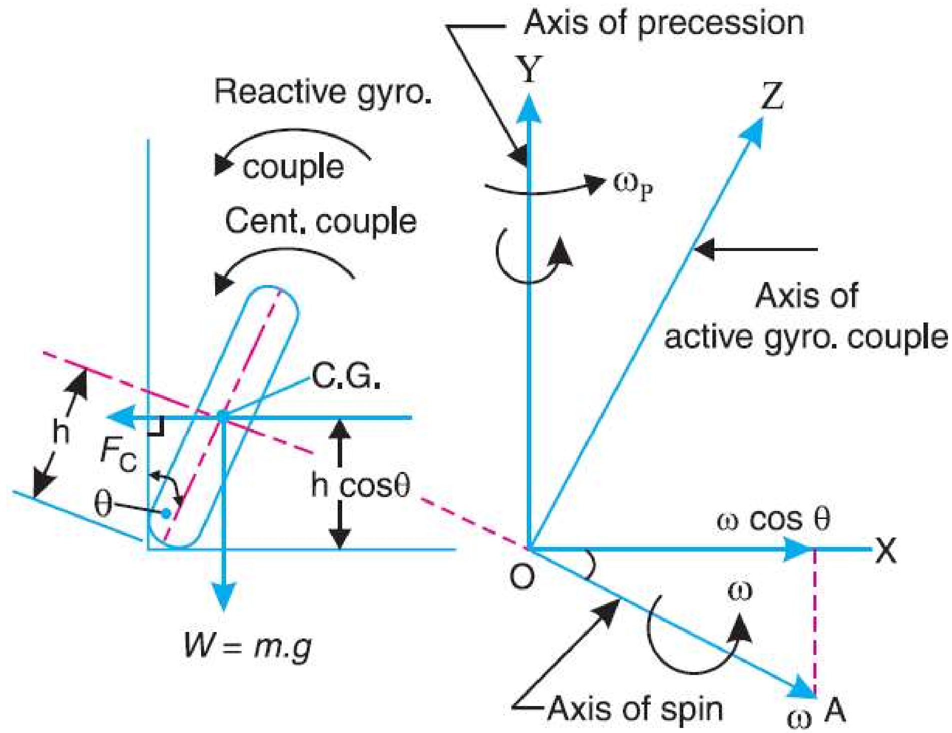
$\therefore$ Gyroscopic couple,

$$C_1 = I.\omega\cos\theta\times\omega_P = \frac{v}{r_W}(2\,I_W \pm G.I_E)\cos\theta\times\frac{v}{R}$$

$$= \frac{v^2}{R.r_W}(2\,I_W \pm G.I_E)\cos\theta$$

Notes:

(a) When the engine is rotating in the same direction as that of wheels, then the positive sign is used in the above expression and if the engine rotates in opposite direction, then negative sign is used.



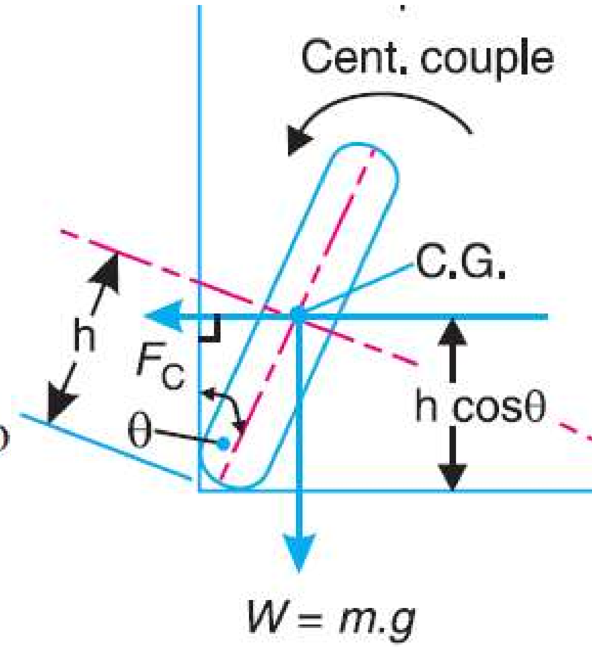
(b) The gyroscopic couple will act over the vehicle outwards i.e. in the anticlockwise direction when seen from the front of the vehicle. The tendency of this couple is to overturn the vehicle in outward direction.

2. *Effect of centrifugal couple*

We know that centrifugal force,

$$F_C = \frac{m.v^2}{R}$$

This force acts horizontally through the centre of gravity ($C.G.$) alo



∴ Centrifugal couple,

$$C_2 = F_C \times h \cos\theta = \left(\frac{m.v^2}{R}\right) h \cos\theta$$

Since the centrifugal couple has a tendency to overturn the vehicle, therefore

Total overturning couple,

$$C_O = \text{Gyroscopic couple} + \text{Centrifugal couple}$$

$$= \frac{v^2}{R.r_W}(2\,I_W + G.I_E)\cos\theta + \frac{m.v^2}{R} \times h\cos\theta$$

$$= \frac{v^2}{R}\left[\frac{2\,I_W + G.I_E}{r_W} + m.h\right]\cos\theta$$

We know that balancing couple = $m.g.h \sin \theta$

The balancing couple acts in clockwise direction when seen from the front of the vehicle. Therefore for stability, the overturning couple must be equal to the balancing couple, *i.e.*

$$\frac{v^2}{R}\left(\frac{2\,I_W + G.I_E}{r_W} + m.h\right)\cos\theta = m.g.h \sin\theta$$

From this expression, the value of the angle of heel (θ) may be determined, so that the vehicle does not skid.

Example 14.15. *Find the angle of inclination with respect to the vertical of a two wheeler negotiating a turn. Given : combined mass of the vehicle with its rider 250 kg ; moment of inertia of the engine flywheel 0.3 kg-m² ; moment of inertia of each road wheel 1 kg-m² ; speed of engine flywheel 5 times that of road wheels and in the same direction ; height of centre of gravity of rider with vehicle 0.6 m ; two wheeler speed 90 km/h ; wheel radius 300 mm ; radius of turn 50 m.*

Solution. Given : $m = 250$ kg ; $I_E = 0.3$ kg-m² ; $I_W = 1$ kg-m² ; $\omega_E = 5\ \omega_W$ or $G = \frac{\omega_E}{\omega_W} = 5$; $h = 0.6$ m ; $v = 90$ km/h $= 25$ m/s ; $r_W = 300$ mm $= 0.3$ m ; $R = 50$ m

We know that gyroscopic couple,

$$C_1 = \frac{v^2}{R \times r_W}(2\ I_W + G.I_E)\cos\theta = \frac{(25)^2}{50 \times 0.3}(2 \times 1 + 5 \times 0.3)\cos\theta$$

$$= 146 \cos\theta \text{ N-m}$$

centrifugal couple,

$$C_2 = \frac{m.v^2}{R} \times h\cos\theta$$

$$= \frac{250\ (25)^2}{50} \times 0.6\cos\theta = 1875\cos\theta \text{ N-m}$$

$\therefore$ Total overturning couple,

$$= C_1 + C_2 = 146 \cos\theta + 1875 \cos\theta = 2021 \cos\theta \text{ N-m}$$

We know that balancing couple

$$= m.g.h \sin\theta = 250 \times 9.81 \times 0.6 \sin\theta = 1471.5 \sin\theta \text{ N-m}$$

Since the overturning couple must be equal to the balancing couple for equilibrium condition, therefore

$$2021 \cos\theta = 1471.5 \sin\theta$$

$$\therefore \quad \tan\theta = \sin\theta / \cos\theta = 2021 / 1471.5 = 1.3734 \text{ or } \theta = 53.94°$$ **Ans.**